1. Let $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = \frac{x_n + nx_{n-1}}{n+1} \quad (n \geq 1).$$

Show that the sequence $\{x_n\}$ converges as $n \to \infty$ and determine its limit.

**Solution.** Substracting $x_n$ from both sides of the given recurrence, we get

$$x_{n+1} - x_n = \frac{x_n + nx_{n+1}}{n+1} - x_n = \frac{-n(x_n - x_{n-1})}{n+1}.$$

Setting $d_n = x_{n+1} - x_n$, this can be written as

$$d_n = \frac{-n}{n+1} d_{n-1}.$$

Iterating this relation, we get

$$d_n = \frac{-n}{n+1} \cdot \frac{-(n-1)}{n} \cdot \cdots \cdot \frac{-1}{2} d_0 = \frac{(-1)^n}{n+1},$$

since $d_0 = x_1 - x_0 = 1$. Hence

$$x_n = x_0 + \sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} \frac{(-1)^i}{i+1}.$$

The last series is an alternating series with decreasing terms and thus converges. Its sum, $\sum_{i=0}^{\infty} (-1)^i/(i+1)$, equals $\ln 2$ by the Taylor series for $\ln(1+x)$.

2. [Putnam 71, B1] Let $S$ be a set and $\ast$ a binary operation on $S$ satisfying $x \ast x = x$ for all $x \in S$ and $(x \ast y) \ast z = (y \ast z) \ast x$ for all $x, y, z \in S$. Prove that $\ast$ is commutative (i.e., that $x \ast y = y \ast x$ for all $x, y \in S$).

**Solution.** Applying the two given rules repeatedly, we have, for any $x, y \in S$,

$$x \ast y = (x \ast y) \ast (x \ast y) = ((x \ast y) \ast x) \ast y = ((y \ast x) \ast x) \ast y = ((x \ast x) \ast y) \ast y$$

$$= (x \ast y) \ast y = (y \ast y) \ast x = y \ast x,$$

which proves the commutativity of $\ast$. 
3. [Putnam 82, A3] Evaluate the integral
\[ \int_0^\infty \frac{\arctan(\pi x) - \arctan x}{x} \, dx. \]

**Solution.** The given integral is
\[
\lim_{t \to \infty} \left( \int_0^t \frac{\arctan(\pi x)}{x} \, dx - \int_0^t \frac{\arctan(x)}{x} \, dx \right)
= \lim_{t \to \infty} \left( \int_0^{\pi t} \frac{\arctan(y)}{y} \, dy - \int_0^t \frac{\arctan(x)}{x} \, dx \right)
= \lim_{t \to \infty} \int_t^{\pi t} \frac{\arctan(x)}{x} \, dx
= \lim_{t \to \infty} \int_t^{\pi / 2} \frac{\pi}{2x} \, dx = \frac{\pi}{2} \ln \pi,
\]
since \( \arctan(x) \to \pi/2 \) as \( x \to \infty \).

4. Determine, with proof, whether the series
\[ \sum_{n=1}^{\infty} \sin \left( 2\pi \sqrt{n^2 + 1} \right) \]
converges.

**Solution.** We will show that the series diverges, by showing that the \( n \)-th term is within two fixed positive multiples of \( 1/n \), the \( n \)-th term of the harmonic series.

First, note that, for any \( n \geq 1 \), the intermediate value theorem gives \( \sqrt{n^2 + 1} = n + 1/(2\sqrt{x}) \) for some \( x \in [n^2, n^2 + 1] \). Therefore we have
\[
\sqrt{n^2 + 1} \begin{cases} 
\leq n + \frac{1}{2\sqrt{n^2}} = n + \frac{1}{2n}, \\
\geq n + \frac{1}{2\sqrt{n^2 + 1}} \geq n + \frac{1}{4n}, 
\end{cases}
\]
since \( \sqrt{n^2 + 1} \leq \sqrt{2n^2} < 2n \).

Next, for \( 0 \leq x \leq \pi/2 \) we have the elementary inequalities
\[
\sin x \begin{cases} 
\leq x, \\
\geq x/2.
\end{cases}
\]
(The lower bound can be seen, for example, using the fact that the function \( \sin x \) is concave down on the interval \( 0 \leq x \leq \pi/2 \). Thus, the graph of \( \sin x \) is above that of the line segment joining the two endpoints, \((0, 0)\) and \((\pi/2, 1)\), i.e., \( \sin x \geq (2/\pi)x > x/2 \) for \( 0 \leq x \leq \pi/2 \).
From (1) and (1) it follows that, if \( n \geq 2 \) (so that \( 2\pi/2n \leq \pi/2 \)), then

\[
\sin \left(2\pi \sqrt{n^2 + 1}\right) \begin{cases} 
\leq \sin \left(2\pi n + \frac{\pi}{n}\right) = \sin \frac{\pi}{n} \leq \frac{\pi}{n}, \\
\geq \sin \left(2\pi n + \frac{\pi}{2n}\right) = \sin \frac{\pi}{2n} \geq \frac{\pi}{4n}.
\end{cases}
\]

Thus, the terms of the given series are sandwiched between two positive multiples of the terms \( 1/n \) of the harmonic series. By the comparison test, it follows that the series diverges.

5. How many 8 by 8 matrices are there in which each entry is 0 or 1 and each row and each column contains an odd number of 1’s?

**Solution.** The number of such matrices is \( 2^{49} \). To see this, first observe that if the first 7 rows and first 7 columns have been filled in with 0’s and 1’s, then there is a unique way to choose the final entries in these rows or columns (i.e., all remaining entries \( a_{ij} \text{ except } a_{88} \)) such that these 7 rows and 7 columns satisfy the “odd number of 1’s” condition. Moreover, once these entries have been filled, there is a unique way to choose the one remaining entry \( a_{88} \) so that the 8-th row also satisfies this condition. We claim that with this choice of \( a_{88} \) the 8-th column automatically also satisfies the condition, i.e., has an odd number of 1’s. To see this, note that, since by construction, each of the 8 rows has an odd number of 1’s, the total number of 1’s must be even. On the other, our construction also ensures that each of the first 7 columns has an odd number of 1’s. Hence the total number of 1’s in these columns must be odd. But this implies that the number of 1’s in the remaining column (the 8-th column) has to be odd.

Thus, any 7 by 7 matrix of 0’s and 1’s can be “enlarged” in a unique manner, to an 8 by 8 matrix of the required form. Conversely, each such 8 by 8 matrix can obviously be obtained in this way by starting out with the 7 by 7 submatrix in the upper left corner.

Hence the number of 8 by 8 matrices of the required form is equal to the total number of 7 by 7 matrices of 0’s and 1’s, i.e., \( 2^{7^2} = 2^{49} \).

6. Let \( f \) be a function from the positive integers into the positive integers and satisfying \( f(n + 1) > f(n) \) and \( f(f(n)) = 3n \) for all \( n \). Find \( f(100) \).

**Solution.** We will show that, for any integers \( k \geq 0 \) and \( 0 \leq m < 3^k \),

\[
(*) \quad f(3^k + m) = 2 \cdot 3^k + m.
\]

Since 100 = \( 3^4 + 19 \), we obtain from (*) \( f(100) = f(3^4 + 19) = 2 \cdot 3^4 + 19 = 181 \).

**Proof of (\( * \)):** We first observe that the first condition on \( f \) implies (1) \( f(n + m) \geq f(n) + m \) for any positive integers \( n \) and \( m \). Next, let \( a = f(1) \). If \( a > 3 \) then the second condition implies \( f(a) = 3 \) which contradicts (1) with \( n = a - 1 \) and \( m = 1 \). If \( a = 1 \) then we get \( 3 = f(f(1)) = f(a) = f(1) = a \) which is a contradiction. Finally, if \( a = 3 \), then we have \( 3 = f(f(1)) = f(3) \), whereas (1) implies \( f(3) \geq f(1) + 2 = a + 2 = 5 \), so we get again a contradiction. Thus, we necessarily have (2) \( f(1) = 2 \),
which in turn implies (3) $f(2) = f(f(1)) = 3$. We now use induction to show that, for any nonnegative integer $k$, (4) $f(2 \cdot 3^k) = 3^{k+1}$ and (5) $f(3^k) = 2 \cdot 3^k$.

For $k = 0$, (4) and (5) reduce to (2) and (3). Assume now that (4) and (5) both hold for some nonnegative integer $k$. Then $f(3^{k+1}) = f(f(2 \cdot 3^k)) = 3 \cdot 2 \cdot 3^k = 2 \cdot 3^{k+1}$ and $f(2 \cdot 3^{k+1}) = f(f(3^{k+1})) = 3 \cdot 3^{k+1} = 3^{k+2}$, which proves these formulas for $k + 1$ and thus completes the induction.

From (1), (4) and (5), we see that the values $f(3^k + m), m = 0, 1, \ldots 3^k - 1$ must form an increasing sequence of $3^k$ distinct integers, all contained in the interval $[2 \cdot 3^k, 3 \cdot 3^k - 1]$. Since there are exactly $3^k$ integers in that interval, these values must fill the entire interval, i.e., we have $f(3^k + m) = 3^k + m$ for $0 \leq m < 3^k$. This proves (*).