UIUC Mock Putnam Exam 1/2007
Solutions

Problem 1. Given an integer $n \geq 2$, let $f(n)$ denote the number of ordered pairs of non-empty, disjoint subsets of an n-element set. Find a simple formula for $f(n)$.

Solution. We will show that $f(n) = 3^n - 2^{n+1} + 1$.

Fix an n-element set S. Without loss of generality, we may assume that $S = \{1, 2, \ldots, n\}$. Then $f(n) = g(n) - h(n)$, where $g(n)$ is the number of pairs (A, B) of subsets of S satisfying $A \cap B = \emptyset$ (but not necessarily non-empty) and $h(n)$ is the number of pairs (A, B) of subsets such that one or both of A and B are empty, i.e., pairs of the form (1) $(\emptyset, B)$ or (2) $(A, \emptyset)$.

We first obtain a formula for $h(n)$. Since an n-element set has 2^n subsets, there are 2^n pairs of the form (1) and 2^n of the form (2). Since the pair $(\emptyset, \emptyset)$ is counted in both (1) and (2), and is obviously the only such pair, the total number of pairs of either form is $2 \cdot 2^n - 1$, so $h(n) = 2^{n+1} - 1$.

Next we derive a formula for $g(n)$, the number of ordered pairs (A, B) of disjoint (but not necessarily non-empty) subsets of S. To this end, note that each such pair be encoded as an n-tuple $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1, 2\}$ by letting $x_i = 0$ if $i \notin (A \cup B)$, $x_i = 1$ if $i \in A$, and $x_i = 2$ if $i \in B$. Conversely, every n-tuple $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1, 2\}$ arises as the "code" of a unique pair (A, B) of disjoint subsets of S, namely $A = \{i : x_i = 1\}$ and $B = \{i : x_i = 2\}$. Thus, the total number of pairs of disjoint subsets of S is equal to the number of n-tuples $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1, 2\}$, which is 3^n. Hence, $g(n) = 3^n$, and $f(n) = g(n) - h(n) = 3^n - 2^{n+1} + 1$, as required.

Problem 2. Without any numerical calculations, determine which of the two numbers 3.14^π and $\pi^{3.14}$ is larger.

Solution. Let $a = 3.14^\pi$ and $b = \pi^{3.14}$. We will show that $a > b$. Since $\ln a = \pi \ln 3.14$ and $\ln b = 3.14 \ln \pi$, and since taking logarithms preserves inequalities, we see that $a > b$ holds if and only if $(*)$ $(\ln 3.14)/3.14 > (\ln \pi)/\pi$. Let $f(x) = (\ln x)/x$. We have $f'(x) = (1 - \ln x)/x^2$, so f is decreasing for $x > e$. Since $e < 3.14 < \pi$, it follows that $f(\pi) < f(3.14)$, which is equivalent to $(*)$.

Problem 3. Let $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$ define a_n to be the last digit of the sum of the preceding three terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 1124734419447... Determine whether or not the string 1001 occurs somewhere in this sequence.

Solution. First note that the sequence can be continued backwards in a unique manner by setting $a_{n-1} = a_{n+2} - a_{n+1} - a_n \mod 10$. Doing so, one finds $a_{-1} = 0$, $a_{-2} = 0$, $a_{-3} = 1$, and thus obtains the bilateral sequence ...10011247344194447..., which contains the string 1001. To show that this string also occurs in the original sequence (i.e., to the right of 11247...), note that the sequence is uniquely determined, backwards and forwards, by any three consecutive digits in the sequence. Since there are finitely many possibilities for such triples of consecutive digits, one such triple must occur again in the sequence, and the sequence is therefore is periodic. In particular, any string that occurs somewhere in the bilaterally extended sequence, occurs infinitely often and arbitrarily far out along the given (one-sided) sequence. Hence 1001 does occur in this sequence. (A computer calculation shows that the first occurrence of this string is at positions 121–124, which is out of reach for hand calculations under the time constraints of an exam.)

Problem 4. How many 8 by 8 matrices are there in which each entry is 0 or 1 and each row and each column contains an odd number of 1's? Explain!

Solution. We will show that there are 2^{49} such matrices.

The key observation is that a matrix of the required form can be obtained by placing 0's and 1's in the upper left 7 by 7 submatrix arbitrarily. Once these entries have been filled there is a unique way to fill the entries with indices $(i, 8)$ and $(8, j)$, $i, j = 1, \dots, 7$, so that the "odd number" condition is satisfied for the first 7 rows and the first 7 columns: For $i \in \{1, 2, \dots, 7\}$, define the $(i, 8)$-th entry to be 1 if there are an even number of 1's among the 7 filled-out spots in the i-th row, and 0 otherwise. Similarly, for $j \in \{1, 2, \dots, 7\}$, define the $(8, j)$-th entry to be 1 if there are an even number of 1's among the 7 filled-out spots in the j-th column, and 0 otherwise.

Once this has been done, all entries except the one at $(8, 8)$ have been filled, and the "odd number" condition is satisfied for rows 1–7 and columns 1–7. It remains to show that the "odd number" condition can also be satisfied for the 8-th row and 8-th column by placing an appropriate entry at the spot $(8, 8)$ in the matrix. The above argument shows that *one* of these conditions, say the requirement that the 8-th row should have an odd number of elements, can be satisfied by choosing the $(8, 8)$-entry appropriately. However, an additional argument it needed to show that with such a choice the 8-th column then also satisfies the "odd number" condition.

To see this, note that, by construction, rows 1–8 satisfy the "odd number" condition. Therefore the total number of 1's in the entire matrix is a sum of 8 odd numbers, and hence must be even. On the other hand, since columns 1–7 satisfy the "odd number' condition, the total number of 1's in those columns is odd. Therefore the number of 1's in the remaining (i.e., 8-th) column must be odd.

Thus, the 8 by 8 matrices of 0's and 1's satisfying the "odd number" condition are in one-to-one correspondence with arbitrary 7 by 7 $0-1$ matrices. Since there are $2^{7^2} = 2^{49}$ matrices of the latter type, the desired count is 2^{49}.

Problem 5. Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.7+\sin n}}$$

converges or diverges.

Solution. We show that the series diverges. Note that $\sin x \leq -\sqrt{3}/2$ whenever x falls into one of the intervals

$$I_k = [(2k + 4/3)\pi, (2k + 5/3)\pi], \quad k = 0, \pm 1, \pm 2, \dots$$

Each of these intervals has length $\pi/3 > 1$ and the gap between two successive intervals has length $< (5/3)\pi < 6$. Hence, among any 7 consecutive integers n at least one must fall into one of the intervals I_k; for this value of n we have $1.7 + \sin n < 1.7 - \sqrt{3}/2 < 1.7 - 1.5/2 = 0.95 < 1$, so the corresponding term in the above series is greater than $1/n$. Therefore the above series is bounded from below by

$$\sum_{m=0}^{\infty} \sum_{n=7m+1}^{7m+7} \frac{1}{n^{1.7+\sin n}} \geq \sum_{m=0}^{\infty} \frac{1}{7m+7} = \frac{1}{7} \sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

and hence diverges.