UIUC Mock Putnam Exam 1/2006
Solutions

Problem 1. Let $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and for $n > 3$ define a_n by $a_n = a_{n-1}a_{n-3}$. Find a_{2006}.

Solution. Computing the first 10 terms of the sequence $\{a_n\}$, we obtain $1, 1, -1, -1, -1, 1, -1, 1, 1, -1$. In particular, we see that the last three terms in this sequence (i.e., a_8, a_9, and a_{10}) are identical to the first three terms. Since for $n > 3$, the n-th term is defined in terms of the three preceding terms, it follows that the sequence repeats itself from the 8th term onwards, i.e., the sequence is periodic with period 7. Since $2006 = 286 \cdot 7 + 4$, the 2006-th term a_{2006} is equal to the 4-th term, i.e., $a_{2006} = a_4 = -1$.

Problem 2. Let F_n denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, and let $S_n = \sum_{k=1}^{n} \frac{F_k}{2^k}$. Find and prove a general formula for S_n.

Solution. Computing the first few terms suggests that

$$S_n = 2 - \frac{F_{n+3}}{2^n}. \qquad (*)$$

We prove this by induction. For $n = 1$, we have $S_1 = 1/2$ and $2 - F_4/2^1 = 2 - 3/2 = 1/2$, so $(*)$ holds in this case. Assume that $(*)$ holds for n. Then

$$S_{n+1} = S_n + \frac{F_{n+1}}{2^{n+1}} = 2 - \frac{F_{n+3}}{2^n} + \frac{F_{n+1}}{2^{n+1}}.$$

Now from the recurrence relation for the Fibonacci numbers we get $F_{n+1} = F_{n+3} - F_{n+2} = F_{n+4} - 2F_{n+2}$ and $F_{n+3} = F_{n+4} - F_{n+2}$. Substituting these expressions for F_{n+1} and F_{n+3} and simplifying gives $(*)$ for $n+1$. Hence, by induction, $(*)$ holds for all n.

Problem 3. Let f be the function defined on all positive integers by $f(1) = 1$ and $f(1) + f(2) + \cdots + f(n) = n^2 f(n)$ for all $n \geq 2$. Find and prove an explicit formula for $f(n)$.

Solution. Subtracting the $(n-1)$st formula from the nth formula, we get $f(n) = n^2 f(n) - (n-1)^2 f(n-1)$ or

$$f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1) \quad (n = 2, 3, \ldots). \tag{1}$$

We now show by induction that $f(n)$ is given by the explicit formula

$$f(n) = \frac{2}{n(n+1)}. \tag{2}$$

Since $f(1) = 1$ by the problem, (2) holds when $n = 1$. Suppose now that (2) holds for some $n \geq 1$. Then, by (1),

$$f(n+1) = \frac{n}{n+2} f(n) = \frac{n}{n+2} \cdot \frac{2}{n(n+1)} = \frac{2}{(n+1)(n+2)},$$

so (2) holds for $n+1$ as well. Hence, by induction, (2) is true for all n.

Problem 4. Determine, with proof, whether or not there exists a real number p with $0 < p < 1$ such that $2005^p + 2007^p = 2 \cdot 2006^p$.

Solution. We claim that there is no such p. Fix p with $0 < p < 1$, and let $f(x) = x^p$. Then the above equality is equivalent to $f(2006) = (1/2)(f(2005)+f(2007))$. Geometrically, this means that the point $(2006, f(2006))$ on the graph of $f(x)$ is exactly the midpoint of the segment joining the two points $(2005, f(2005))$ and $(2007, f(2007))$. However, for $0 < p < 1$, this is impossible, since the function $f(x)$ is concave down for such p (since $f''(x) = p(p-1)x^{p-2} < 0$).

Problem 5. Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k$.

Solution. Without the factor $(-1)^k$, the sum would be equal to $\binom{2n}{n}$, by a familiar binomial identity (the Vandermonde identity). The combinatorial proof of this identity doesn't (easily) generalize to the present situation, but the generating function approach can be adapted:

Writing $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$, we see that the sum of the problem is the coefficient of x^n in the product

$$\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \right) \left(\sum_{m=0}^{n} \binom{n}{m} x^m \right),$$

which equals $(1-x)^n(1+x)^n$ by the binomial theorem. But $(1-x)^n(1+x)^n == (1-x^2)^n$, and expanding the latter expression by the binomial theorem, we obtain

$$(1-x^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{2k}.$$

The coefficient of x^n here, and therefore the binomial sum to be evaluated, is equal to 0 when n is odd (since the sum involves only even powers of x), and $(-1)^{n/2}\binom{n}{n/2}$ when n is even.