UIUC Mock Putnam Exam 1/2006

(Solutions at **`www.math.uiuc.edu/contests.html`**)

Problem 1. Let $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and for $n > 3$ define a_n by $a_n = a_{n-1}a_{n-3}$. Find a_{2006}.

Problem 2. Let F_n denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, and let $S_n = \sum_{k=1}^{n} \frac{F_k}{2^k}$. Find and prove a general formula for S_n.

Problem 3. Let f be the function defined on all positive integers by $f(1) = 1$ and $f(1) + f(2) + \cdots + f(n) = n^2 f(n)$ for all $n \geq 2$. Find and prove an explicit formula for $f(n)$.

Problem 4. Determine, with proof, whether or not there exists a real number p with $0 < p < 1$ such that $2005^p + 2007^p = 2 \cdot 2006^p$.

Problem 5. Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k$.