Instructions

Rules
The rules are the same as for the Putnam. No calculators, books, notes, etc. Show all work, but do not hand in scratch work. Blank sheets of paper for scratch work will be provided. Make sure to cross out anything you do not want to be considered for grading.

Grading
If you are doing the exam as a take-home exam, drop it off at my office, 241 Illini Hall, or leave it in my mailbox in 250 Altgeld for grading. Problems will be graded in much the same way as the Putnam exam problems. Each problem is worth 10 points. Partial credit may be given, but only if significant progress towards a solution is being made. As in the Putnam exam, a clear, logical write-up of the solution is essential. All work must be shown. An answer alone, without justification, doesn’t count. Graded exams will be returned in the next training section.

Solutions
Exam solutions will be posted on the UIUC Math Contest Web Page within a few days.

Suggested time limit
1.5 hours. If you take the contest at home, time yourself.
Problem 1. Let $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and for $n > 3$ define $a_n$ by $a_n = a_{n-1}a_{n-3}$. Find $a_{2006}$. 
Problem 2. Let $F_n$ denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, and let $S_n = \sum_{k=1}^{n} \frac{F_k}{2^k}$. Find and prove a general formula for $S_n$. 
Problem 3. Let $f$ be the function defined on all positive integers by $f(1) = 1$ and $f(1) + f(2) + \cdots + f(n) = n^2 f(n)$ for all $n \geq 2$. Find and prove an explicit formula for $f(n)$. 
Problem 4. Determine, with proof, whether or not there exists a real number $p$ with $0 < p < 1$ such that $2005^p + 2007^p = 2 \cdot 2006^p$. 
Problem 5. Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k$. 
