UIUC Mock Putnam Exam 1/2005
Solutions

Problem 1. **Without any numerical calculations**, determine which of the two numbers e^π and π^e is larger.

Solution. Let $a = e^\pi$ and $b = \pi^e$. We will show that $a > b$. Since $\ln a = \pi \ln e = \pi$ and $\ln b = e \ln \pi$, and since taking logarithms preserves inequalities, we see that $a > b$ holds if and only if $(*)$ $(\ln e)/e > (\ln \pi)/\pi$. Now consider the function $f(x) = (\ln x)/x$. We have $f'(x) = (1 - \ln x)/x^2$, so f is decreasing for $x > e$, and since $\pi > e$, it follows that $f(\pi) < f(e)$ which is equivalent to $(*)$.

Problem 2. Prove that, given any set of 10 distinct positive integers below 100, there exist two *disjoint* non-empty subsets whose elements have the same sum.

Solution. Let A be a set of 10 distinct positive integers ≤ 100. Note that the sum over the elements of any subset of A must be between 1 and $10 \cdot 100 = 1000$, so there are at most $10 \cdot 1000$ possible values for such a sum. On the other hand, there are $2^{10} - 1 = 1023$ nonempty subsets of A. Hence, by the pigeonhole principle, there exist two distinct nonempty subsets $B, C \subset A$ whose elements have the same sum. If B and C are disjoint, we are done. Otherwise, let $B' = B \setminus B \cap C$, $C' = C \setminus B \cap C$. Then, by construction, B' and C' are disjoint. Moreover, B' and C' still have the same element sum since they were obtained from B and C by removing the same set of elements. Finally, each of B' and C' is nonempty, since if they were both empty, then we would have $B = C$, contradicting our assumption that B and C are distinct, and if one of B' and C' is empty and the other non-empty, then B' and C' could not have the same element sum.

Problem 3. How many positive integers are there which, in their decimal representation, have strictly decreasing digits? Explain!

Solution. The key to this problem is the fact that there is a one-to-one correspondence between *nonnegative* integers (i.e., positive or zero) with decreasing digits and nonempty subsets of $\{0, 1, \dots, 9\}$. This is seen as follows: First, given such a subset $\{a_1, \dots, a_k\}$, with the a_i's written in decreasing order, let n be the integer obtained by concatenating the a_i's, i.e., $n = a_1 a_2 \dots a_k$. Then n is a nonnegative integer with decreasing digits. Conversely, given a nonnegative integer with decreasing digits, let $a_1, a_2, \dots, a_k$ denote the sequence of its digits. Then the set $\{a_1, \dots, a_k\}$ is a nonempty subset of $\{0, 1, \dots, 9\}$.

Thus, the number of nonnegative integers with decreasing digits is equal to the number of non-empty subsets of the 10-element set $\{0, 1, \dots, 9\}$, i.e., $2^{10} - 1 = 1023$. The integer 0 is included in this count, so the number of *positive* integers with decreasing digits sought in the problem is one less, i.e., 1022.

Problem 4. Let $x = 0.0102040816326428\dots$ be the real number in the interval $(0, 1)$ whose decimal expansion (after the decimal period) is obtained by concatenating the last two digits of the sequence of powers of 2 (padded by 0 in case there is only one digit). Is x is rational? Explain!

Solution. We claim that x is rational. To show this, it suffices to show that its decimal expansion is ultimately periodic. Now, by definition, $x = 0.a_0 a_1 a_2 \dots$, where a_n denotes the last two digits of 2^n, padded with zeros when necessary. Thus, it suffices to show that the sequence $\{a_n\}_{n=0}^{\infty}$ is ultimately periodic.

To this end, first note that a_n is the remainder of 2^n upon division by 100, i.e., $a_n \equiv 2^n \bmod 100$. Since there are only finitely many possible values for a_n, there exist $m < n$ such that $a_m = a_n$. Now, if $a_m = a_n$, then

$$2^{n+i} = 2^i \cdot 2^n \equiv 2^i a_n = 2^i a_m \equiv 2^i \cdot 2^m = 2^{m+i} \bmod 100,$$

and hence $a_{n+i} = a_{m+i}$ for all $i \geq 1$. Thus, the sequence a_n is ultimately periodic (with period $n - m$). This proves our claim and completes the proof.

Problem 5. Find a function $f(x)$ satisfying

$$f(x) + f\left(\frac{1}{1-x}\right) = x$$

for all $x \neq 0, 1$.

Solution. Let (1) denote the given equation for $f(x)$. Substituting $(1-x)^{-1}$ for x in (1), we get a second equation,

$$f\left(\frac{1}{1-x}\right) + f\left(1 - \frac{1}{x}\right) = \frac{1}{1-x}, \tag{2}$$

valid for all $x \neq 0, 1$. We repeat the process one more time, substituting $(1-x)^{-1}$ for x in (2). The result is

$$f\left(1 - \frac{1}{x}\right) + f(x) = 1 - \frac{1}{x}. \tag{3}$$

We stop here, since the next substitution would only lead back to the original equation (1). However, we now have enough equations to determine $f(x)$: Setting $X = f(x)$, $Y = f(1/(1-x)$, and $Z = f(1-1/x))$, the three equations (1), (2), (3), can be written as

$$\begin{aligned} X + Y &= x, \\ Y + Z &= \frac{1}{1-x}, \\ Z + X &= 1 - \frac{1}{x}. \end{aligned}$$

This is a linear system of three equations in three unknowns X, Y, Z, which is easily solved. The result is

$$X = f(x) = \frac{1}{2}\left(x - \frac{1}{1-x} + 1 - \frac{1}{x}\right).$$

One can check that this function is indeed a solution to (1).

Problem 6. Let A denote the set of positive integers whose decimal expansion does not have a zero. Determine, with proof, whether the infinite series $\sum_{n \in A} 1/n$ converges.

Solution. We break the range of summation into finite intervals $I_k = 10^{k-1} \leq n < 10^k$, $k = 1, 2, \ldots$, and let $S_k = \sum_{n \in I_k \cap A} 1/n$ denote the corresponding partial sum. Now, note that an integer $n \in I_k$ has at most k digits, and if n is also in the set A, then there are only 9 possible values for each of these digits. Hence the total number of elements in $I_k \cap A$ is at most 9^k. On the other hand, if $n \in I_k$ then $n \geq 10^{k-1}$. Thus,

$$S_k = \sum_{n \in I_k \cap A} \frac{1}{10^{k-1}} \leq \frac{1}{n} \#\{n : n \in I_k \cap A\} \leq \frac{9^k}{10^{k-1}},$$

and therefore

$$\sum_{n \in A} \frac{1}{n} = \sum_{k=1}^{\infty} S_k \leq 10 \sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^k = \frac{9}{1 - 9/10} = 90 < \infty.$$

Hence the series converges.

Problem 7. Let $P(t)$ be a non-constant polynomial with real coefficients. Prove that there exist only finitely many positive real numbers x such that

$$\int_0^x P(t)\sin t dt = \int_0^x P(t)\cos t dt = 0.$$

Solution. Let $f(x) = \int_0^x P(t)e^{it}dt$. Then $f(x) = 0$ if and only if the two integrals $\int_0^x P(t)\sin t dt$ and $\int_0^x P(t)\cos t dt$ are simultaneously zero. By induction, one can show that for each nonnegative integer k, $\int_0^x t^k e^{it}dt = Q_k(x)e^{ix} + c_k$, where $Q_k(x)$ is a polynomial of degree k with real coefficients and c_k a constant. It follows that, if $P(t) = \sum_{k=0}^n a_k t^k$ is a polynomial of degree n with real coefficients, then $f(x) = Q(x)e^{ix} + c$ where $Q(x) = \sum_{k=0}^n a_k Q_k(x)$ is a polynomial of the same degree as $P(t)$ with real coefficients and $c = \sum_{k=0}^n a_k c_k$ a constant. Setting $x = 0$, we see that c must be equal to $Q(0)$. Thus, $f(x) = 0$ is equivalent to $(*)$ $Q(x) = -Q(0)e^{-ix}$, and we have to show that $(*)$ has only finitely many solutions.

By our assumption, the polynomial $P(x)$ is non-constant. Since $Q(x)$ has the same degree as $P(x)$, $Q(x)$ is also non-constant. Hence $|Q(x)| \to \infty$ if $|x| \to \infty$. Therefore, all solutions to $(*)$ must fall into an interval of the form $|x| \leq C$, where C is a constant.

If $Q(0) = 0$, then $(*)$ reduces to the equation $Q(x) = 0$ which has at most n solutions by the fundamental theorem of algebra, and we are done. Suppose therefore that $Q(0) \neq 0$. Taking real parts, we obtain from $(*)$ $Q(x) = -Q(0)\cos x$. If this equation had infinitely many solutions in the interval $[-C, C]$, then by applying Rolle's theorem to the function $Q(x) + Q(0)\cos x$ and its successive derivatives, we obtain that, for each positive integer k, the equation $(**)$ $Q^{(4k)}(x) = -Q(0)\cos x$, also had infinitely many solutions x in the same interval. (Note that Rolle's theorem does not hold for complex-valued functions; thus it was necessary to take real parts in $(*)$ before applying the theorem.) However, since $Q(x)$ is a polynomial of degree n, the left side of $(**)$ is zero if $k > n/4$, whereas (since $Q(0) \neq 0$) the right side can only be zero at the zeroes of $\cos x$, of which there are only finitely many in the interval $[-C, C]$. Hence $(*)$ can have only finitely many solutions.

Remark. The last part of the solution can be considerably shortened with a bit of complex analysis magic. Namely, let $f(z) = Q(z) + Q(0)e^{-iz}$. Since $Q(z)$ is a non-constant polynomial, this function is an analytic function of z that is not constant. By a theorem from complex analysis, such a function can have at most finitely many zeros in any disk $|z| \leq C$. In particular, it can have at most finitely many *real* zeros in this disk, and this is exactly what we showed above using more elementary methods.