UIUC Mock Putnam Exam 1/2005

Problem 1. **Without any numerical calculations**, determine which of the two numbers e^π and π^e is larger.

Problem 2. Prove that, given any set of 10 distinct positive integers below 100, there exist two *disjoint* non-empty subsets whose elements have the same sum.

Problem 3. How many positive integers are there which, in their decimal representation, have strictly decreasing digits? Explain!

Problem 4. Let $x = 0.0102040816326428\ldots$ be the real number in the interval $(0, 1)$ whose decimal expansion (after the decimal period) is obtained by concatenating the last two digits of the sequence of powers of 2 (padded by 0 in case there is only one digit). Is x is rational? Explain!

Problem 5. Find a function $f(x)$ satisfying

$$f(x) + f\left(\frac{1}{1-x}\right) = x$$

for all $x \neq 0, 1$.

Problem 6. Let A denote the set of positive integers whose decimal expansion does not have a zero. Determine, with proof, whether the infinite series $\sum_{n\in A} 1/n$ converges.

Problem 7. Let $P(t)$ be a non-constant polynomial with real coefficients. Prove that there exist only finitely many positive real numbers x such that

$$\int_0^x P(t)\sin t dt = \int_0^x P(t)\cos t dt = 0.$$