Problem 1. Let \( f(x) = x^3e^{x^2}(1 - x^2)^{-1} \). Find \( f^{(7)}(0) \), the 7th derivative of \( f \) at 0. (No brute force calculations!)

Problem 2. Does there exist a multiple of 2004 whose decimal representation involves only a single digit? Explain!

Problem 3. Evaluate the sum \( \sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k \).

Problem 4. Let \( f \) be a real-valued continuously differentiable function on the interval \([0, 1]\) with \( \int_0^1 f(x) \, dx = 0 \). Show that
\[
\int_0^1 f(x)^2 \, dx \leq \frac{1}{2} \int_0^1 |f'(x)| \, dx \int_0^1 |f(x)| \, dx.
\]