Problem 1. Show that $\log_{2002} 2003 + \log_{2003} 2002 > 2$.

Solution. Setting $a = \log_{2002} 2003$ and $b = \log_{2003} 2002$, where $\log$ denotes the natural logarithm, the inequality to be proved becomes $(a/b) + (b/a) > 2$. Since $a$ and $b$ are positive, this is equivalent to $a^2 + b^2 > 2ab$, and the latter inequality holds since $0 < (a - b)^2 = a^2 + b^2 - 2ab$.

Problem 2. Show that the number

$$\frac{2^3}{3^2} + \frac{4^3}{5^2} + \frac{6^3}{7^2} + \cdots + \frac{2002^3}{2003^2}$$

is not an integer.

Solution. The key observation here is that 2003 is a prime number. If we write the sum as a single fraction, the common denominator will involve a factor 2003, whereas the numerator is a sum of terms all of which, except for the last one, are divisible by 2003. Hence, after cancelling any common (prime) factors, there remains a factor 2003 in the denominator. Thus the given sum cannot be an integer.

Problem 3. Find the sum of the infinite series

$$\sum_{n=3}^{\infty} \frac{1}{\binom{n}{3}}.$$

Solution. The sum is equal to $3/2$. To see this, use the identity

$$\frac{2}{n(n-1)(n-2)} = \frac{1}{(n-1)(n-2)} - \frac{1}{n(n-1)}$$

to convert the sum into one that “telescopes”:

$$\sum_{n=3}^{\infty} \frac{1}{\binom{n}{3}} = \sum_{n=3}^{\infty} \frac{6}{n(n-1)(n-2)} = \sum_{n=3}^{\infty} \left( \frac{3}{(n-1)(n-2)} - \frac{3}{n(n-1)} \right)$$

$$= \sum_{m=2}^{\infty} \frac{3}{m(m-1)} - \sum_{n=3}^{\infty} \frac{3}{n(n-1)} = \frac{3}{2}.$$

(Note here that splitting the infinite series into two series is permissible since the individual series are both absolutely convergent. However, this would no
longer be the case if one would go one step further and write, for example, 
$1/n(n-1) = 1/(n-1) - 1/n$, since then the resulting individual series would 
be divergent.

**Problem 4.** Find all positive integers $n$ which are equal to the sum of 
digits of $n^2$.

**Solution.** We will show that $n = 1$ and $n = 9$ are the only such integers. 
To see this, suppose that $n$ has $k$ digits. Then, on the one hand $n \geq 10^{k-1}$, 
and on the other hand $n^2 < 10^{2k}$, so the sum of digits of $n^2$ is at most 
$2 \cdot 9k = 18k$. Thus if $n$ is a solution to the problem, then $10^{k-1} \leq 18k$, 
which implies $k \leq 2$. (One way to see this is by noting that the function 
$f(x) = 10^{x-1}/(18x)$ is $> 1$ at $x = 3$ and is increasing for $x > 3$.) Therefore 
$n < 10^2$, so $n^2 < 10^4$, and hence the sum of digits of $n^2$ is at most $4 \cdot 9 = 36$. 
Thus any solution $n$ is at most 36 (in fact, strictly less than 36 since 9999 
is not a square). To further reduce the number of cases to be checked, we 
use the fact that the sum of digits of a number is congruent to this number 
modulo 9. Thus, any solution $n$ to the problem must satisfy 
$n \equiv n^2 \pmod{9}$, 
and this is only possible if $n \equiv 1 \pmod{9}$ or $n \equiv 0 \pmod{9}$. Thus, among 
the integers less than 36 only $n = 1, 9, 10, 18, 19, 27, 28$ remain as possible 
solutions. Checking these numbers directly then easily shows that only $n = 1$ 
and $n = 9$ are solutions.

**Problem 5.** Is there a prime number $p$ and a sequence $(P_n)_{n \in \mathbb{N}}$ of integer 
points (i.e., points with integer coordinates) in the plane such that every 
integer point in the plane belongs to the sequence $(P_n)_{n \in \mathbb{N}}$ and the distance 
between any two consecutive points $P_n$ and $P_{n+1}$ equals $p$?

**Solution.** The answer is “yes”; for example, we can take $p = 5$. To prove 
this, first observe that it is enough to show that we can get from the origin to 
the point $(1,0)$. For suppose we have a path of the desired type from $(0,0)$ 
to $(1,0)$. By rotating this path by 90, 180, or 270 degrees, we can then also 
reach the points $(0,1), (-1,0), (0,-1)$ from the origin. By repeated 
applications of such moves, we reach any point with integer coordinates.

Now, the path $(0,0) \rightarrow (-5,0) \rightarrow (-2,4) \rightarrow (1,0)$ is a path of the desired 
form, i.e., it leads from the origin to the point $(1,0)$ via integer points, in 
moves of length 5.

**Further remarks.** Clearly $p = 2$ will not work, because then the only 
possible movements are horizontal or vertical steps of length 2, and so if 
$P_0$ is the origin, say, then the only possible coordinates of each $P_n$ will be 
even integers. Similarly for $p = 3$ only horizontal or vertical moves in steps 
of 3 are possible, so starting at the origin we can only reach points whose 
coordinates are multiples of 3. The key property of the prime $p = 5$ is 
that it is equal to the diagonal distance between two integer points, namely 
$5 = \sqrt{3^2 + 4^2}$. This enables diagonal moves of length 5. More generally, it is 
known from number theory that any prime $p \equiv 1 \pmod{4}$ has this property.