2021 UI FRESHMAN MATH CONTEST
October 30, 2021, 10 am – 12 pm

Solutions

1. (a) Find 4 distinct integers a_1, a_2, a_3, a_4, such that the 6 pairwise sums $a_i + a_j$, $1 \le i < j \le 4$, when written in increasing order, represent 6 consecutive integers.

   (b) Prove that there do **not** exist 5 distinct integers a_1, a_2, a_3, a_4, a_5, such that the 10 pairwise sums $a_i + a_j$, $1 \le i < j \le 5$, when written in increasing order, represent 10 consecutive integers.

Solution. (a) The integers $1, 2, 3, 5$ have this property: Their pairwise sums are $1 + 2 = 3$, $1 + 3 = 4$, $1 + 5 = 6$, $2 + 3 = 5$, $2 + 5 = 7$, $3 + 5 = 8$, which represent the integers $3, 4, \ldots, 8$.
(b) The sum of 10 consecutive integers $n, n + 1, \ldots, n + 9$ is

$$\sum_{i=0}^{9}(n+i) = 10n + \frac{10 \cdot 9}{2} = 10n + 45.$$

On the other hand, since each of the five numbers $a_1, \ldots, a_5$ occurs in exactly 4 pairwise sums, the sum of these pairwise sums must be equal to $4(a_1 + \cdots + a_5)$. But this is an even number, so it cannot equal $10n + 45$.

2. Evaluate the integral

$$\int_0^\infty \frac{x-1}{1+x^3}\,dx.$$

Solution. The answer is $\boxed{0}$. For the proof, split the given integral into $I_1 + I_2$, where

$$I_1 = \int_0^1 \frac{x-1}{1+x^3}\,dx, \quad I_2 = \int_1^\infty \frac{x-1}{1+x^3}\,dx. \tag{1}$$

Substituting $y = 1/x$, $dy = -(1/x^2)dx = -y^2 dx$ in I_2, we obtain

$$I_2 = \int_1^0 \frac{y^{-1}-1}{1+y^{-3}}(-y^{-2})dy = \int_0^1 \frac{1-y}{y^3+1}dy = -I_1.$$

Thus, the given integral is $I_1 + I_2 = I_1 + (-I_1) = 0$.

Alternate solution. Factor the denominator as $1 + x^3 = (1 + x)(1 - x + x^2)$ and expand the integrand into partial fractions:

$$\frac{x-1}{1+x^3} = \frac{x-1}{(1+x)(1-x+x^2)} = \frac{1}{3}\left(\frac{-2}{x+1} + \frac{2x-1}{x^2-x+1}\right).$$

Then the given integral becomes

$$\begin{aligned}
\int_0^\infty \frac{x-1}{1+x^3}dx &= \lim_{t\to\infty}\int_0^t \frac{x-1}{1+x^3}dx = \lim_{t\to\infty}\left(-\frac{2}{3}\int_0^t \frac{1}{x+1}dx + \frac{1}{3}\int_0^t \frac{2x-1}{x^2-x+1}dx\right) \\
&= \lim_{t\to\infty}\left(-\frac{2}{3}\ln(t+1) + \frac{1}{3}\ln(t^2-t+1)\right) \\
&= \lim_{t\to\infty}\frac{1}{3}\left(\ln\frac{t^2-t+1}{(t+1)^2}\right) = \lim_{t\to\infty}\frac{1}{3}\left(\ln\frac{1-(1/t)+(1/t^2)}{(1+(1/t))^2}\right) = \frac{1}{3}\ln 1 = 0.
\end{aligned} \tag{2}$$

Note: In the second solution one cannot directly split up the given infinite integral into a sum over the integrals $\frac{-2}{3}\int_0^\infty \frac{1}{x+1}dx$ and $\frac{1}{3}\int_0^\infty \frac{2x-1}{x^2-x+1}dx$ since the latter integrals both diverge. Thus, one must perform this splitting on the finite integral over $[0, t]$, and then take the limit as $t \to \infty$.

In the first solution this issue does not arise since the integral I_1 in (1) is an integral over a finite interval and the integral I_2 converges absolutely. Thus there is no need to work with finite integrals and then take the limit.

3. Consider a right triangle with sides a and b and hypothenuse $c = \sqrt{a^2 + b^2}$. Denote by S the set of all points in the interior of the triangle which are closer to the hypothenuse than to each of the other two sides. Find, with proof, the area of S.

Solution. The answer is $\boxed{\dfrac{abc}{2(a+b+c)}}$.

For the proof, let A, B, C be the vertices opposite the sides a, b, c, respectively. Let O be the point at which the bisectors of the three angles meet. Then the set S is the triangle $\Delta(AOB)$.

From elementary trigonometry we know that O is the center of the inscribed circle of the triangle $\Delta(ABC)$. Let r denote the radius of this circle. Then

$$\text{Area}(S) = \text{Area}(\Delta(AOB)) = \frac{1}{2}rc, \quad \text{Area}(\Delta(BOC)) = \frac{1}{2}ra, \quad \text{Area}(\Delta(AOC)) = \frac{1}{2}rb.$$

and

$$\text{Area}(\Delta(ABC)) = \text{Area}(\Delta(AOB)) + \text{Area}(\Delta(BOC)) + \text{Area}(\Delta(AOC)) = \frac{r}{2}(a+b+c).$$

On the other hand, since $\Delta(ABC)$ is a right triangle with sides a and b, we also have $\text{Area}(\Delta(ABC)) = ab/2$. Therefore, $r = ab/(a+b+c)$ and

$$\text{Area}(S) = \frac{1}{2}rc = \frac{abc}{2(a+b+c)}.$$

4. Given a positive real number x, define a double sequence $x_{m,n}$, $m, n \geq 0$, by

$$x_{m,0} = \frac{x}{2^m}, \quad m = 0, 1, \ldots$$
$$x_{m,n+1} = x_{m,n}^2 + 2x_{m,n}, \quad m = 0, 1, \ldots, \quad n = 0, 1, \ldots$$

(a) Find, with proof, a simple general formula for $x_{m,n}$.

(b) Show that the diagonal sequence $x_{n,n}$, $n = 0, 1, \ldots$, converges and find the limit $\lim_{n\to\infty} x_{n,n}$.

Solution. (a) Set $y_{m,n} = x_{m,n} + 1$. Then the given recurrence is equivalent to

$$y_{m,0} = 1 + \frac{x}{2^m}, \quad m = 0, 1, \ldots$$
$$y_{m,n+1} = y_{m,n}^2, \quad m = 0, 1, \ldots, \quad n = 0, 1, \ldots$$

Iterating the latter relation, we get

$$y_{m,n} = y_{m,n-1}^2 = \cdots = y_{m,0}^{2^n} = \left(1 + \frac{x}{2^m}\right)^{2^n}$$

and therefore

(1) $$\boxed{x_{m,n} = \left(1 + \frac{x}{2^m}\right)^{2^n} - 1}$$

This is the desired general formula for $x_{m,n}$.

(b) We will show that $\lim_{n\to]infty} x_{n,n} = \boxed{e^x - 1}$.

For the proof, note that from (1) we have

$$x_{n,n} = \left(1 + \frac{x}{2^n}\right)^{2^n} - 1.$$

Since $\lim_{n\to\infty}(1 + x/n)^n = e^x$, it follows that the limit $\lim_{n\to\infty} x_{n,n}$ exists and is equal to $e^x - 1$.

5. Determine, with proof, the number of subsets of the set $\{1, 2, \ldots, 30\}$ with the property that the sum of all elements in the subset is greater than 232.

Solution. We will show that there are $\boxed{2^{29}}$ such subsets.

The key observations are that (i) 232.5 is exactly half the sum of all elements in the set $\{1, 2, \ldots, 30\}$, namely $\sum_{k=1}^{30} k = 30(30+1)/2 = 465$, and (ii) if a subset $S \subset \{1, 2, \ldots, 30\}$ has sum s, then its complement, S^c, has sum $465 - s$.

It follows from these observations that the mapping $S \to S^c$ is a one-to-one correspondence between subsets with sum $> 465/2 = 232.5$ and subsets with sum $< 465/2 = 232.5$. Hence exactly half of the 2^{30} subsets of $\{1, \ldots, 30\}$ have sum > 232.5, and the other half have sum < 232.5.