2021 UI FRESHMAN MATH CONTEST
October 30, 2021, 10 am – 12 pm

1. (a) Find 4 distinct integers a_1, a_2, a_3, a_4, such that the 6 pairwise sums $a_i + a_j$, $1 \leq i < j \leq 4$, when written in increasing order, represent 6 consecutive integers.

   (b) Prove that there do **not** exist 5 distinct integers a_1, a_2, a_3, a_4, a_5, such that the 10 pairwise sums $a_i + a_j$, $1 \leq i < j \leq 5$, when written in increasing order, represent 10 consecutive integers.

2. Evaluate the integral
$$\int_0^\infty \frac{x-1}{1+x^3}\,dx.$$

3. Consider a right triangle with sides a and b and hypothenuse $c = \sqrt{a^2+b^2}$. Denote by S the set of all points in the interior of the triangle which are closer to the hypothenuse than to each of the other two sides. Find, with proof, the area of S.

4. Given a positive real number x, define a double sequence $x_{m,n}$, $m, n \geq 0$, by
$$x_{m,0} = \frac{x}{2^m}, \quad m = 0, 1, \ldots$$
$$x_{m,n+1} = x_{m,n}^2 + 2x_{m,n}, \quad m = 0, 1, \ldots, \quad n = 0, 1, \ldots$$

   (a) Find, with proof, a simple general formula for $x_{m,n}$.

   (b) Show that the diagonal sequence $x_{n,n}$, $n = 0, 1, \ldots$, converges and find the limit $\lim_{n\to\infty} x_{n,n}$.

5. Determine, with proof, the number of subsets of the set $\{1, 2, \ldots, 30\}$ with the property that the sum of all elements in the subset is greater than 232.