2020 UI FRESHMAN MATH CONTEST
October 17, 2020
Solutions

1. Let

$$S(n) = \sum_{k=1}^{n} \frac{1}{\langle \sqrt{k} \rangle},$$

where $\langle x \rangle$ denotes the integer closest to x, with the convention that when x is exactly between two integers, then x is rounded *up* instead of down. Thus, for example, $\langle 1.73 \rangle = 2$, $\langle 2.5 \rangle = 3$, and $\langle 3.14159 \rangle = 3$.
Find and prove a general formula for $S(m^2)$, where m is a positive integer.

Solution. We will show by induction that, for all positive integers m, $S(m^2) = 2m - 1$.
In the base case $m = 1$, we have $S(1^2) = 1/1 = 1 = 2 \cdot 1 - 1$, so the claimed formula holds. For the induction step suppose the formula holds for some $m \geq 1$. Note that, since $(m + 1/2)^2 = m^2 + m + 1/4$, we have $\langle \sqrt{k} \rangle = m$ if $m^2 < k \leq m^2 + m$ and $\langle \sqrt{k} \rangle = m + 1$ if $m^2 + m + 1 \leq k \leq m^2 + 2m + 1 = (m+1)^2$. Therefore

$$\begin{aligned} S((m+1)^2) - S(m^2) &= \sum_{k=m^2+1}^{m^2+m} \frac{1}{m} + \sum_{k=m^2+m+1}^{m^2+2m+1} \frac{1}{m+1} \\ &= \frac{1}{m} \cdot m + \frac{1}{m+1} \cdot (m+1) = 2. \end{aligned}$$

and hence $S((m+1)^2) = S(m^2) + 2 = 2m - 1 + 2 = 2m + 1 = 2(m+1) - 1$. This proves the desired formula for $m + 1$ and completes the induction.

2. Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, \dots\}$ be the set of integers of the form $2^a 3^b$, where a, b are nonnegative integers. Prove that, given any 5 distinct integers in S, there exist two of these 5 integers whose product is a square. For example, if the five integers are $2, 3, 6, 8, 36$, then $2 \cdot 8$ is a square.

Solution. Write the five integers as $n_i = 2^{a_i} 3^{b_i}$, $i = 1, \dots, 5$, and consider the parities (even or odd) of the exponent pairs (a_i, b_i). Since there are only four possible pairs of parities ((even,even), (even,odd), (odd,even), and (odd,odd)), by the pigeonhole principle there exists distinct indices i and j such that (a_i, b_i) and (a_j, b_j) have the same pair of parities. But then $a_i + a_j$ and $b_i + b_j$ are both even, i.e, of the form $a_i + a_j = 2h$ and $b_i + b_j = 2k$, where h and k are nonnegative integers. It follows that $n_i n_j = 2^{a_i + a_j} 3^{b_i + b_j} = 2^{2h} 3^{2k} = (2^h 3^k)^2$ is a square.

3. Let $P(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + 1$ be a polynomial of degree $n \geq 2$ such that the coefficients $a_1, \dots, a_{n-1}$ are *positive* real numbers and all roots of $P(x)$ are real (they may be repeated).
(a) Prove that $P(2) \geq 3^n$.
(b) For each integer $n \geq 2$ find a polynomial $P_n(x)$ of the above form that achieves this lower bound, i.e., satisfies $P_n(2) = 3^n$.

Solution. (a) Since $P(x)$ has positive coefficients and only real roots, as positive coefficients and only real roots, its roots must be negative real numbers, say $-c_1, \dots, -c_n$. Then, since the leading coefficient of $P(x)$ is 1, $P(x)$ factors as $P(x) = \prod_{i=1}^{n} (x + c_i)$. By the AGM Inequality, we have

$$2 + c_i = 3 \frac{1 + 1 + c_i}{3} \geq 3 \left(1 \cdot 1 \cdot c_i\right)^{1/3} = 3 c_i^{1/3}$$

for each i, and hence

$$P(2) = \prod_{i=1}^{n}(2 + c_i) \geq 3^n \left(\prod_{i=1}^{n} c_i\right)^{1/3} = 3^n P(0)^{1/3} = 3^n,$$

since $P(0) = a_0 = 1$.
(b) The bound 3^n is achieved by the polynomial $P_n(x) = \sum_{k=0}^{n}\binom{n}{k}x^k = (1 + x)^n$, which is of the desired form with root -1, repeated n times.

4. A computer generates three random real numbers, x, y, z, in the interval $[0, 1]$, then computes the sum of these numbers, $s = x + y + z$, and outputs $\langle s\rangle$, the integer that is closest to s (with the convention that numbers that lie exactly between two integers will be rounded *up*). Thus, the output, $\langle s\rangle$, will be of the four numbers 0, 1, 2, and 3. Find, with proof, the probability that $\langle s\rangle = 1$.

Solution. We claim that the desired probability is $23/48$.
Let $P(k)$ denote the probability that the rounded sum is k. Note that if (x, y, z) is a triple of numbers of $[0, 1]$ with rounded sum $\langle x + y + z\rangle = k$, then the "complementary" triple $(1 - x, 1 - y, 1 - z)$ has rounded sum $3 - k$. It follows that $P(0) = P(3)$ and $P(1) = P(2)$. Since $\sum_{k=0}^{3} P(k) = 1$, this implies that $P(0) + P(1) = 1/2$, and hence $P(1) = 1/2 - P(0)$.
Next note that $\langle x + y + z\rangle = 0$ if and only if $x + y + z < 1/2$. Thus $P(0)$ is the volume of the part of the unit cube that lies below the plane $x + y + z = 1/2$. But this region is a right-angled tetrahedron with sides $1/2$, so its volume is $(1/6)(1/2)^3 = 1/48$. Hence $P(0) = 1/48$ and $P(1) = 1/2 - 1/48 = 23/48$.

5. Find, with proof, all positive rational solutions of the equation

$$(x + y)^y = x^{x+y}. \tag{1}$$

Solution. We will show that the positive rational solutions to (1) are those of the form

$$(x, y) = ((1 + z)^z, (1 + z)^z z), \quad z \in \mathbb{N}. \tag{2}$$

Suppose first that (x, y) is of the form (2). Then x and y are positive rational numbers (in fact, positive integers) and we have

$$\begin{aligned} x + y &= (1 + z)^z(1 + z) = (1 + z)^{1+z}, \\ (x + y)^y &= (1 + z)^{(1+z)z(1+z)^z} = (1 + z)^{z(1+z)^{1+z}}, \\ x^{x+y} &= (1 + z)^{z(1+z)^{1+z}} = (x + y)^y. \end{aligned}$$

Thus, any pair (x, y) of the form (2) is a solution to (1).
Conversely, suppose that (x, y) are positive rational numbers satisfying (1). Setting $y = zx$, we can rewrite (1) as $x^{zx}(1 + z)^{zx} = x^{x+zx}$, or equivalently

$$(1 + z)^z = x. \tag{3}$$

Write $x = m/n$ and $z = p/q$, where (m, n) and (p, q) are coprime pairs of positive integers. Then, by (3), $(1 + p/q)^{p/q} = m/n$, or equivalently

$$n^q(p + q)^p = m^q q^p. \tag{4}$$

Since n and m have no common prime factor, (4) implies that n^q must divide q^p. On the other hand, since p and q have no common prime factor, $p + q$ and q have no common prime factor either, and so from (4) we conclude that q^p must divide n^q. It follows that $q^p = n^q$.

We claim that this is only possible if $q = 1$. Indeed, if $q > 1$, then there exists a prime factor $p_0 \geq 2$ that divides q. Since $q^p = n^q$, p_0 must also divide n. Let a be the exponent of p_0 in q and b be the exponent of p_0. Then ap is the exponent of p_0 in q^p and bq is the exponent of p_0 in n^q, so we must have $ap = bq$. But since p and q are coprime, q must divide a, and hence $q \leq a$. But then $q \geq p_0^a \geq 2^a \geq 2^q$, which is a contradiction since $2^q > q$ for all positive integers q.

Thus, we have shown that if (2) holds, then $q = 1$, so $z = p/q = p$ is a positive integer. By (3) it follows that $x = (1+z)^z$ and $y = zx = z(1+z)^z$, i.e., (x, y) is of the form (2).