2020 UI FRESHMAN MATH CONTEST
October 17, 2020

1. Let
$$S(n) = \sum_{k=1}^{n} \frac{1}{\langle\sqrt{k}\rangle},$$
where $\langle x\rangle$ denotes the integer closest to x, with the convention that when x is exactly between two integers, then x is rounded *up* instead of down. Thus, for example, $\langle 1.73\rangle = 2$, $\langle 2.5\rangle = 3$, and $\langle 3.14159\rangle = 3$.
Find and prove a general formula for $S(m^2)$, where m is a positive integer.

2. Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, \dots\}$ be the set of integers of the form $2^a 3^b$, where a, b are nonnegative integers. Prove that, given any 5 distinct integers in S, there exist two of these 5 integers whose product is a square. For example, if the five integers are $2, 3, 6, 8, 36$, then $2 \cdot 8$ is a square.

3. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ be a polynomial of degree $n \geq 2$ such that the coefficients $a_1, \dots, a_{n-1}$ are *positive* real numbers and all roots of $P(x)$ are real (they may be repeated).
(a) Prove that $P(2) \geq 3^n$.
(b) For each integer $n \geq 2$ find a polynomial $P_n(x)$ of the above form that achieves this lower bound, i.e., satisfies $P_n(2) = 3^n$.

4. A computer generates three random real numbers, x, y, z, in the interval $[0, 1]$, then computes the sum of these numbers, $s = x + y + z$, and outputs $\langle s\rangle$, the integer that is closest to s (with the convention that numbers that lie exactly between two integers will be rounded *up*). Thus, the output, $\langle s\rangle$, will be of the four numbers 0, 1, 2, and 3. Find, with proof, the probability that $\langle s\rangle = 1$.

5. Find, with proof, all positive rational solutions of the equation

(1) $$(x + y)^y = x^{x+y}.$$