1. Let

\[ S(n) = \sum_{k=1}^{n} \frac{1}{\langle \sqrt{k} \rangle}, \]

where \( \langle x \rangle \) denotes the integer closest to \( x \), with the convention that when \( x \) is exactly between two integers, then \( x \) is rounded up instead of down. Thus, for example, \( \langle 1.73 \rangle = 2 \), \( \langle 2.5 \rangle = 3 \), and \( \langle 3.14159 \rangle = 3 \).

Find and prove a general formula for \( S(m^2) \), where \( m \) is a positive integer.

2. Let \( S = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, \ldots \} \) be the set of integers of the form \( 2^a 3^b \), where \( a, b \) are nonnegative integers. Prove that, given any 5 distinct integers in \( S \), there exist two of these 5 integers whose product is a square. For example, if the five integers are 2, 3, 6, 8, 36, then \( 2 \cdot 8 \) is a square.

3. Let \( P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1 \) be a polynomial of degree \( n \geq 2 \) such that the coefficients \( a_1, \ldots, a_{n-1} \) are positive real numbers and all roots of \( P(x) \) are real (they may be repeated).
   (a) Prove that \( P(2) \geq 3^n \).
   (b) For each integer \( n \geq 2 \) find a polynomial \( P_n(x) \) of the above form that achieves this lower bound, i.e., satisfies \( P_n(2) = 3^n \).

4. A computer generates three random real numbers, \( x, y, z \), in the interval \([0, 1]\), then computes the sum of these numbers, \( s = x + y + z \), and outputs \( \langle s \rangle \), the integer that is closest to \( s \) (with the convention that numbers that lie exactly between two integers will be rounded up). Thus, the output, \( \langle s \rangle \), will be of the four numbers 0, 1, 2, and 3. Find, with proof, the probability that \( \langle s \rangle = 1 \).

5. Find, with proof, all positive rational solutions of the equation

\[ (x + y)^y = x^{x+y}. \]