2019 UI FRESHMAN MATH CONTEST
October 12, 2019, 10 am – 12 pm
Solutions

1. Which positive integers n satisfy the inequality $n^{n+1} < (n+1)^n$? Explain your reasoning!

   Solution. Answer: $\boxed{1, 2}$
   Proof: Since $1^2 < 2^1$ and $2^3 < 3^2$, the given inequality holds for $n = 1$ and $n = 2$.

   Now let $n \geq 3$. We will show that the inequality does not hold in this case. Taking logarithms, we see that the given inequality holds if and only if $(n+1)\ln n < n \ln(n+1)$, which in turn is equivalent to $(*)$ $(\ln n)/n < (\ln(n+1))/(n+1)$.

   The inequality $(*)$ is of the form $(**)$ $f(n) < f(n+1)$, where $f(x) = (\ln x)/x$. Now note that $f'(x) = (1 - \ln x)/x^2$ is negative if $x > e = 2.78\ldots$, so in particular, $f(x)$ is decreasing for all $x \geq 3$. It follows that $(**)$ does not hold if $n \geq 3$. This completes the proof.

2. Let α be a real number with $0 < \alpha < 1$. If two points are selected randomly from the interval $[0, 1]$, what is the probability that the distance between them is at least α? Justify your answer.

   Solution. (B2, Putnam 1961). Answer: $\boxed{(1-\alpha)^2}$
   Proof: Picking two points x and y randomly from the interval $[0, 1]$ is equivalent to picking a two-dimensional point (x, y) from the unit square $[0, 1] \times [0, 1]$. The probability that x and y have distance at least α is then given by the area of the region inside the unit square for which $|x - y| \geq \alpha$. This region consists of two right triangles of sides $1 - \alpha$ each, so its area is $(1 - \alpha)^2$, as claimed.

3. Determine, with proof, the number of ways to write 2019 in the form $2019 = \sum_{i=1}^{k} a_i$, where k is an arbitrary positive integer and the numbers a_i are positive integers satisfying $a_1 \leq \cdots \leq a_k \leq a_1 + 1$. (Examples of representations of the required form are $2019 = 1 + 1 + \cdots + 1$ (with 2019 terms 1), $2019 = 2 + 2 + \cdots + 2 + 3 + 3 + 3$ (with 1005 terms 2 and 3 terms 3), $2019 = 1009 + 1010$, and $2019 = 2019$.)

   Solution. (A1, Putnam 2003) Answer: $\boxed{2019}$
   Proof: Clearly there can be at most 2019 terms in such a representation, i.e., we must have $1 \leq k \leq 2019$. We will show that, for each such k, there exists exactly one representation of the required form.

   Let $k \in \{1, 2, \ldots, 2019\}$ be given. To prove the existence of a representation with k terms, use division by remainder to get $2019 = ka + r$, where $a \in \mathbb{N}$ and $0 \leq r < k$. Then setting $a_i = a$ for $i = 1, \ldots, k-r$, and $a_i = a+1$ for $i = k-r+1, \ldots, k$, we have $(*)$ $a_1 + \cdots + a_k = (k-r)a + r(a+1) = ka + r = 2019$, so we obtain a representation of the required form with k terms.

   To show that this is the only such representation involving k terms, let $2019 = a_1 + \cdots + a_k$ be an arbitrary representation with k terms that satisfies the given conditions. Then this representation must either contain k identical terms a_1, or, for some positive integer $r < k$, exactly $k - r$ terms a_1 and r terms $a_1 + 1$. In either case we have $2019 = (k - r)a_1 + r(a_1 + 1) = ka_1 + r$, where $0 \leq r < k$, so the representation must be of the form $(*)$.

   Thus, for each $k = 1, 2, \ldots, 2019$ there exists exactly one representation of 2019 of the desired form involving k terms. Hence the total number of such representations is 2019, as claimed.

4. For $n = 1, 2, \dots$ let
$$f_n(x) = \prod_{k=0}^{n-1} \left(1 + x^{2^k}\right).$$
Find a simple general formula for $f_n(x)$, valid for any $x > 1$ and any positive integer n.

Solution. Answer: $\boxed{f_n(x) = (x^{2^n} - 1)/(x - 1)}$
Proof: Expanding the product gives
$$f_n(x) = \sum_{I \subset \{1,\dots,n-1\}} x^{\sum_{i \in I} 2^i},$$
where I runs over all subsets of $\{0, 1, \dots, n-1\}$ (including the empty subset). Now observe that, as I runs through these subsets, the exponents $\sum_{i \in I} 2^i$ run through the integers $0, 1, \dots, 2^n - 1$, with each of these integers being "hit" exactly once. Thus, we have
$$f_n(x) = x^0 + \cdots + x^{2^n - 1} = \frac{x^{2^n} - 1}{x - 1},$$
by the formula for the sum of a finite geometric series. This proves our claim.

5. Call a pair (a, b) of positive integers "good" if a and b have the same prime factors. For example, the pair $(12, 54)$ is good since $12 = 2^2 \cdot 3$ and $54 = 2 \cdot 3^3$, while $(12, 16)$ is not good since 16 does not have 3 as prime factor. Find, with proof, an infinite sequence of pairs (a, b), with $a \neq b$, such that both (a, b) and $(a - 1, b - 1)$ are good.

Solution. We claim that, for any positive integer n, the pair $(a_n, b_n) = (2^n - 1, (2^n - 1)^2)$ has the desired property.
Since $a_n^2 = b_n$, it is obvious that each pair (a_n, b_n) itself is good. Moreover, we have $a_n - 1 = 2^n - 2$ and $b_n - 1 = (2^n - 1)^2 - 1 = (2^n - 1 + 1)(2^n - 1 - 1) = 2^n(2^n - 2) = 2^n(a_n - 1)$. Thus, $a_n - 1$ and $b_n - 1$ have exactly the same prime factors except possibly for the prime factor 2. However, since both $a_n - 1$ and $b_n - 1$ are even, they both contain 2 as prime factor, so the pair $(a_n - 1, b_n - 1)$ is good as well. This completes the proof.

6. Prove or disprove: There exists a sequence $a_1, a_2, \dots$ of positive integers satisfying the following properties for each $k \in \mathbb{N}$:

(i) a_k has exactly k digits in its decimal representation.
(ii) a_{k+1} is obtained from a_k by attaching either the digit 1 or the digit 2 to the **left** of a_k.
(iii) a_k is divisible by 2^k.

(An example of a sequence satisfying (i) and (ii) is $2, 12, 112, 2112, 12112, \dots$, though this sequence does not satisfy (iii) for every k.)

Solution. We use induction on k to prove the existence of such a sequence. For $k = 1$, $a_1 = 2$ satisfies conditions (i)–(iii).
Now let $k \geq 1$, and suppose we have constructed a number a_k with k decimal digits, each 1 or 2, that is divisible by 2^k. Considering congruences modulo 2^{k+1} we then have either (I) $a_k \equiv 0 \bmod 2^{k+1}$ or (II) $a_k \equiv 2^k \bmod 2^{k+1}$.
In case (I), we construct a_{k+1} by adding the digit 2 to the left of a_k. Then $a_{k+1} = 2 \cdot 10^k + a_k = 2^{k+1}5^k + a_k \equiv a_k \equiv 0 \bmod 2^{k+1}$, so a_{k+1} is divisible by 2^{k+1} .
In case (II), we construct a_{k+1} by adding the digit 1 to the left of a_k. Then $a_{k+1} = 1 \cdot 10^k + a_k = 2^k 5^k + a_k \equiv (5^k + 1)2^k \bmod 2^{k+1} \equiv 0 \bmod 2^{k+1}$, so as before a_{k+1} is divisible by 2^{k+1}.
Thus, in either case a_{k+1} satisfies all three conditions (i)—(iii) and the induction is complete.