2019 UI FRESHMAN MATH CONTEST
October 12, 2019, 10 am – 12 pm

1. Which positive integers n satisfy the inequality $n^{n+1} < (n+1)^n$? Explain your reasoning!

2. Let α be a real number with $0 < \alpha < 1$. If two points are selected randomly from the interval $[0, 1]$, what is the probability that the distance between them is at least α? Justify your answer.

3. Determine, with proof, the number of ways to write 2019 in the form $2019 = \sum_{i=1}^{k} a_i$, where k is an arbitrary positive integer and the numbers a_i are positive integers satisfying $a_1 \le \cdots \le a_k \le a_1 + 1$. (Examples of representations of the required form are $2019 = 1 + 1 + \cdots + 1$ (with 2019 terms 1), $2019 = 2 + 2 + \cdots + 2 + 3 + 3 + 3$ (with 1005 terms 2 and 3 terms 3), $2019 = 1009 + 1010$, and $2019 = 2019$.)

4. For $n = 1, 2, \ldots$ let
$$f_n(x) = \prod_{k=0}^{n-1} \left(1 + x^{2^k}\right).$$
Find a simple general formula for $f_n(x)$, valid for any $x > 1$ and any positive integer n.

5. Call a pair (a, b) of positive integers "good" if a and b have the same prime factors. For example, the pair $(12, 54)$ is good since $12 = 2^2 \cdot 3$ and $54 = 2 \cdot 3^3$, while $(12, 16)$ is not good since 16 does not have 3 as prime factor. Find, with proof, an infinite sequence of pairs (a, b), with $a \ne b$, such that both (a, b) and $(a - 1, b - 1)$ are good.

6. Prove or disprove: There exists a sequence $a_1, a_2, \ldots$ of positive integers satisfying the following properties for each $k \in \mathbb{N}$:

   (i) a_k has exactly k digits in its decimal representation.

   (ii) a_{k+1} is obtained from a_k by attaching either the digit 1 or the digit 2 to the **left** of a_k.

   (iii) a_k is divisible by 2^k.

   (An example of a sequence satisfying (i) and (ii) is $2, 12, 112, 2112, 12112, \ldots$, though this sequence does not satisfy (iii) for every k.)