2018 UI FRESHMAN MATH CONTEST
October 13, 2018, 10 am – 12 pm

Solutions

1. Find a function $f(x)$ such that $f(f(x)) = 2018x + 2017$ for all real numbers x or show that no such function exists.

   Solution. Trying a linear function $f(x) = ax + b$, we get $f(f(x)) = a(ax + b) + b = a^2x + (a + 1)b$. If we set $a = \sqrt{2018}$ and $b = \sqrt{2018} - 1$, this becomes $f(f(x)) = 2018x + 2017$. Thus the function $\boxed{f(x) = \sqrt{2018}x + \sqrt{2018} - 1}$ satisfies the given relation.

2. Consider the equation

   (1) $$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0,$$

   where each a_i is either $+1$ or -1.

   (a) Find a sequence $a_i = \pm 1$, $i = 1, \ldots, 2016$, that satisfies (1) for $n = 2016$.

   (b) Prove that, when $n = 2018$, (1) has no solution in integers $a_i = \pm 1$.

   Solution. More generally, we will show that (1) has a solution if and only if n is divisible by 4. Since 2016 is divisible by 4, while 2018 is not divisible by 4, this gives the desired result.

   (a) Suppose that n is divisible by 4. Let $a_1, a_2, \ldots, a_n$ be the pattern $(1, 1, -1, -1)$ repeated $n/4$ times. Then the n terms a_ia_{i+1}, $i = 1, \ldots, n$, in (1) are alternately 1 and -1. Since n is even it follows that their sum is equal to 0. Thus, (1) has a solution whenever n is divisible by 4.

   (b) Now suppose that n is not divisible by 4. If n is odd, then the left-hand side of (1) consists of a sum of an odd number of terms, each ± 1, and thus cannot be equal to 0.
   It n is even, but not divisible by 4, then n is of the form $n = 2m$, where m is odd. Suppose that there exist integers $a_i = \pm 1$, $i = 1, 2, \ldots, n$ satisfying (1). Then exactly m of the terms $a_1a_2, \ldots, a_na_1$ must be equal to -1, and m terms must be equal to 1. Therefore the product of all $2m$ terms is $(-1)^m 1^m = -1$ since m is odd. On the other hand, directly multiplying the terms shows that this product is $(a_1a_2)(a_2a_3)\ldots(a_na_1) = a_1^2a_2^2\ldots a_n^2 = 1$, so we have reached a contradiction. Thus, no solution to (1) exists when n is not divisible by 4.

3. Let $s(n)$ denote the sum of the decimal digits of an integer n. For example, $s(2018) = 2 + 0 + 1 + 8 = 11$. Find the smallest positive integer n such that $s(n)$ and $s(n + 1)$ are both divisible by 17 or show that no such integer exists.

   Solution. We claim that $\boxed{n = 8899}$ is the smallest such number.
   Note that $s(8899) = 34 = 2 \cdot 17$ and $s(8900) = 17$, so 8899 has the desired divisibility property.
   It remains to show that there is no smaller number with this property. It suffices to consider four-digit integers of the form $(*)$ $abcd$, where $a, b, c, d \in \{0, 1, \ldots, 9\}$ and $a \leq 8$. (We allow leading digits 0.)
   Note first that we must have $d = 9$ in $(*)$ since if $d < 9$, then $s(n + 1) = s(n) + 1$, so $s(n)$ and $s(n + 1)$ cannot both be divisible by 17. Similarly, if $d = 9$, but $c < 9$, then $n + 1$ becomes $ab(c + 1)0$, so $s(n + 1) = a + b + c + 1 + 0 = s(n) - 8$, and $s(n)$ and $s(n + 1)$ cannot both be divisible by 17.
   Thus, we must have $c = d = 9$ in $(*)$, so our number is of the form $ab99$, where $a \in \{0, 1, \ldots, 8\}$ and $b \in \{0, 1, \ldots, 9\}$. Then $s(n) = a + b + 18 \equiv a + b + 1 \mod 17$, so to satisfy $s(n) \equiv 0 \mod 17$, we need $a + b \equiv 16 \mod 17$. Since a and b are decimal digits with $a \leq 8$, the only choice for (a, b) is $(a, b) = (8, 8)$, which corresponds to the integer 8899 obtained above. Thus, 8899 is the smallest integer with the desired property.

4. Determine the set of nonnegative integers that can be written as $\lfloor x\rfloor+\lfloor 2x\rfloor+\lfloor 4x\rfloor$ for some positive real number x. ($\lfloor x\rfloor$ is the floor function, defined as the largest integer $\le x$.)

Solution. Let $s(x)=\lfloor x\rfloor+\lfloor 2x\rfloor+\lfloor 4x\rfloor$, and let $S=\{s(x): x>0\}$ be the set of numbers of the desired form. We claim that S is the set of integers $n\ge 0$ satisfying $\boxed{n\equiv 0,1,3,4 \mod 7}$.

For the proof, first note that, for any nonnegative integer k, $s(x+k)=\lfloor x+k\rfloor+\lfloor 2x+2k\rfloor+\lfloor 4x+4k\rfloor = s(x)+7k$. Thus, the set S is periodic modulo 7, and to prove the claim it suffices to show that the elements of S in $\{0,1,\dots,6\}$ are exactly the numbers $0,1,3,4$. This follows from the following case-by-case analysis of the values of $s(x)$ for $0<x<1$.

- If $0<x<1/4$, then $s(x)=0+0+0=0$.
- If $1/4\le x<1/2$, then $s(x)=0+0+1=1$.
- If $1/2\le x<3/4$, then $s(x)=0+1+2=3$.
- If $3/4\le x<1$, then $s(x)=0+1+3=4$.

5. Given a finite nonempty set A of real numbers, let $P(A)$ denote the product of all elements of A. For example, $P(\{2,5,6\})=2\cdot 5\cdot 6=60$. Set $P(\emptyset)=1$ ($\emptyset$ denotes the empty set). Evaluate, with proof, the sum

$$\sum_{A\subset\{1,2,\dots,2018\}}\frac{1}{P(A)},$$

where A runs over all subsets of the set $\{1,2,\dots,2018\}$ (including the empty subset).

Solution. The given sum is S_{2018}, where

$$S_n=\sum_{A\subset\{1,\dots,n\}}\frac{1}{P(A)},$$

We will show that $(*)$ $S_n=n+1$ for all positive integers n. Taking $n=2018$, we obtain $\boxed{2019}$ as the answer to the given problem.

For the proof of $(*)$, we consider the product

$$P_n=\prod_{k=1}^{n}\left(1+\frac{1}{k}\right).$$

Multiplying out the right side gives a sum over 2^n terms, with one term being 1 and the other 2^n-1 terms being of the form $1/(k_1\dots k_r)$ where $\{k_1,\dots,k_r\}$ is a nonempty subset of $\{1,\dots,n\}$. Conversely, every non-empty subset of $\{1,\dots,n\}$ contributes exactly one term $1/(k_1\dots k_r)$ to this expansion. Thus, we have $S_n=P_n$. On the other hand, we have

$$P_n=\prod_{k=1}^{n}\frac{k+1}{k}=n+1,$$

since the product telescopes. Hence $S_n=P_n=n+1$ for all n, as claimed.

6. Define a sequence $a_1<a_2<a_3<\cdots$ of positive integers as follows. Let $a_1=1$ and for each $n\ge 2$ let a_n be the smallest integer greater than a_{n-1} that is not of the form $2a_k$ for some $k<n$. The first few terms of this sequence are $1,3,4,5,7,9,11,12,13,15,16,17,19,20,\dots$.

How many terms of this sequence are ≤ 2048? Justify your answer.

Solution. The answer is $\boxed{1365}$.

For the proof, let $A(N)$ denote the number of terms a_i satisfying $a_i\le N$. We will show that, for any positive integer n, $(*)$ $A(2\cdot 4^n)=(4^{n+1}-1)/3$. With $n=5$, this gives the above answer: $A(2048)=(4^6-1)/3=1365$.

To prove $(*)$ observe that the sequence $\{a_i\}$ consists of exactly those positive integers that have an even power of 2 in their prime factorization, i.e., integers of the form $2^{2k}m$, where m is odd and $k=0,1,\dots$.

For given k, the number of positive integers $\le 2\cdot 4^n$ that are of the form $2^{2k}m$ with m odd is equal to the number of odd positive integers $\le 2\cdot 4^n/4^k$, which is 4^{n-k}.

Summing over $k=0,\dots,n$, we get $A(2\cdot 4^n)=\sum_{k=0}^{n}4^{n-k}=(4^{n+1}-1)/3$ as claimed.