2018 UI FRESHMAN MATH CONTEST
October 13, 2018, 10 am – 12 pm

1. Find a function $f(x)$ such that $f(f(x)) = 2018x + 2017$ for all real numbers x or show that no such function exists.

2. Consider the equation

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0, \tag{1}$$

where each a_i is either $+1$ or -1.

   (a) Find a sequence $a_i = \pm 1$, $i = 1, \ldots, 2016$, that satisfies (1) for $n = 2016$.

   (b) Prove that, when $n = 2018$, (1) has no solution in integers $a_i = \pm 1$.

3. Let $s(n)$ denote the sum of the decimal digits of an integer n. For example, $s(2018) = 2 + 0 + 1 + 8 = 11$. Find the smallest positive integer n such that $s(n)$ and $s(n + 1)$ are both divisible by 17 or show that no such integer exists.

4. Determine the set of nonnegative integers that can be written as $\lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 4x\rfloor$ for some positive real number x. ($\lfloor x\rfloor$ is the floor function, defined as the largest integer $\le x$.)

5. Given a finite nonempty set A of real numbers, let $P(A)$ denote the product of all elements of A. For example, $P(\{2, 5, 6\}) = 2 \cdot 5 \cdot 6 = 60$. Set $P(\emptyset) = 1$ ($\emptyset$ denotes the empty set). Evaluate, with proof, the sum

$$\sum_{A \subset \{1,2,\ldots,2018\}} \frac{1}{P(A)},$$

where A runs over all subsets of the set $\{1, 2, \ldots, 2018\}$ (including the empty subset).

6. Define a sequence $a_1 < a_2 < a_3 < \cdots$ of positive integers as follows. Let $a_1 = 1$ and for each $n \ge 2$ let a_n be the smallest integer greater than a_{n-1} that is not of the form $2a_k$ for some $k < n$. The first few terms of this sequence are $1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, \ldots$.

   How many terms of this sequence are ≤ 2048? Justify your answer.