2017 UI FRESHMAN MATH CONTEST
September 16, 2017, 10 am – 12 pm
Solutions

1. Suppose a is a nonzero real number such that $a + \dfrac{1}{a}$ is an integer.

   (a) Prove that $a^2 + \dfrac{1}{a^2}$ is an integer.

   (b) Prove that for any positive integer n, $a^n + \dfrac{1}{a^n}$ is an integer.

   Solution.

   (a) The identity

   $$a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$$

   shows that if $a + 1/a$ is an integer, then so is $a^2 + 1/a^2$.

   (b) (The following elegant argument is due to Alan Hu. An alternative, but more complicated, approach is to expand $(a + 1/a)^n$ by the Binomial Theorem.) Let

   $$A_n = a^n + \frac{1}{a^n}.$$

   We will use strong induction on n to prove that A_n is an integer for all integers $n \geq 0$. Clearly $A_0 = 2$ is an integer, while $A_1 = a + 1/a$ is an integer by hypothesis. Now let $n \geq 2$ and assume that $A_0, A_1, \ldots, A_{n-1}$ are integers. We want to show that A_n is an integer as well. We have

   $$\begin{aligned} A_1 A_{n-1} &= \left(a + \frac{1}{a}\right)\left(a^{n-1} + \frac{1}{a^{n-1}}\right) \\ &= a^n + a^{n-2} + \frac{1}{a^{n-2}} + \frac{1}{a^n} \\ &= A_n + A_{n-2}. \end{aligned}$$

   Hence $A_n = A_1 A_{n-1} - A_{n-2}$, and since A_1, A_{n-1}, and A_n are all integers, so is A_n. This completes the induction.

2. The number $10! = 3628800$ ends in 2 zeros, $20! = 2432902008176640000$ ends in 4 zeros, and $40! = 8 \ldots 72000000000$ (a 48 digit number) ends in 9 zeros. How many zeros are at the end of the number $1000!$ (a number with 2568 digits)? Justify your answer.

   Solution. We claim that $1000!$ ends in exactly $\boxed{249}$ zeros.
   To prove this, it suffices to show that $k = 249$ is the largest power such that both 2^k and 5^k divide N. Since $1000! = 1 \cdot 2 \cdot 3 \cdots \cdot 1000$ contains at least 500 even factors, it is clearly divisible by 2^{249}. Thus, it remains to show that the maximal power of 5 dividing $1000!$ is 5^{249}.
   Now of the 1000 factors $1, 2, \ldots, 1000$, exactly $1000/5 = 200$ are divisible by 5 and each of these contributes (at least) one factor 5 to $1000!$. Moreover, $1000/5^2 = 40$ of these factors are divisible by 5^2 and thus contribute (at least) one additional factor 5 to $1000!$. Similarly, $\lfloor 1000/5^3 \rfloor = 8$ factors are divisible by 5^3 and contribute a third factor 5 to $1000!$, $\lfloor 1000/5^4 \rfloor = 1$ factors are divisible by 5^4 and contribute a fourth factor 5 to $1000!$, while none of the factors is divisible by 5^5 (or a higher power of 5). Thus, the total number of factors 5 dividing $1000!$ is

   $$\lfloor \frac{1000}{5} \rfloor + \lfloor \frac{1000}{25} \rfloor + \lfloor \frac{1000}{125} \rfloor + \lfloor \frac{1000}{625} \rfloor = 200 + 40 + 8 + 1 = 249.$$

3. Let a, b, c be real numbers > 1, and let

$$S = \log_a(bc) + \log_b(ca) + \log_c(ab),$$

where $\log_b x$ denotes the base b logarithm of x. Find, with proof, the smallest possible value of S.

Solution. Setting $A = \log a$, $B = \log b$, $C = \log c$, we have $A, B, C > 0$ (since $a, b, c > 1$) and the sum S becomes

$$S = \frac{B}{A} + \frac{C}{A} + \frac{C}{B} + \frac{A}{B} + \frac{A}{C} + \frac{B}{C}.$$

By the arithmetic-geometric mean inequality, this sum is

$$\geq 6 \left(\sqrt{\frac{B \cdot C \cdot C \cdot A \cdot A \cdot B}{A \cdot A \cdot B \cdot B \cdot C \cdot C}} \right)^{1/6} = 6.$$

Hence $S \geq 6$. The example $A = B = C = 1$ shows that this bound is attained, so 6 is the smallest possible value of S.

4. Given a positive integer n, let $a_n = \lfloor \sqrt{n} \rfloor$ denote the greatest integer less than or equal to $\sqrt{n}$; for example, $a_5 = \lfloor \sqrt{5} \rfloor = 2$, and $a_{12} = \lfloor \sqrt{12} \rfloor = 3$. How many positive integers $n \leq 10^6$ are divisible by a_n? Justify your answer.

Solution. We claim that there are exactly $\boxed{2998}$ integers with the given property.
For the proof, note that if $m = \lfloor \sqrt{n} \rfloor$, then n must be in the range $m^2 \leq n \leq (m+1)^2 - 1 = m^2 + 2m$. Among these integers n, exactly three, namely $n = m^2, m^2 + m, m^2 + 2m$, are divisible by m. For $m = 1, 2 \ldots, 999$, all three of these n-values are $\leq 10^6$, while for $m = 1000$, only $n = m^2$ is in this range. Hence there are $3 \cdot 999 + 1 = 2998$ positive integers $n \leq 10^6$ that are divisible by a_n.

5. Let a, b, c, d be positive integers such that $ab = cd$. Show that $a^2 + b^2 + c^2 + d^2$ is a composite number.

Solution. The condition $ab = cd$ implies that $a/c = d/b$. Let x/y be the reduced form of this fraction (so that x and y are integers with no common prime factor). Then $a = xh$, $c = yh$, $d = yk$, $b = yk$, for some positive integers k and y. Therefore

$$a^2 + b^2 + c^2 + d^2 = x^2h^2 + y^2k^2 + y^2h^2 + y^2k^2 = (h^2 + k^2)(x^2 + y^2),$$

which is composite since each factor is an integer ≥ 2.

6. Define a sequence of positive integers $a_0, a_1, a_2, \ldots$ by $a_0 = 1$ and

$$a_{n+1} = \begin{cases} \frac{1}{2} a_n & \text{if } a_n \text{ is even,} \\ \frac{1}{2}(a_n + 2017) & \text{if } a_n \text{ is odd,} \end{cases}$$

for $n = 0, 1, 2, \ldots$.

(a) Prove that the sequence $\{a_n\}$ is periodic.

(b) Find, with proof, a_{2017}.

Solution.

(a) First we show by induction that all terms a_n are positive integers satisfying $a_n < 2017$ for all $n \geq 0$. Since $a_0 = 1$, this clearly holds for $n = 0$. Now let $n \geq 0$ and suppose that a_n is a positive integer satisfying $a_n < 2017$. Then

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n < \frac{1}{2} \cdot 2017 < 2017 & \text{if } a_n \text{ is even,} \\ \frac{1}{2}(a_n + 2017) < \frac{1}{2}(2017 + 2017) < 2017 & \text{if } a_n \text{ is odd,} \end{cases}$$

so in both cases a_{n+1} is a positive integer satisfying $a_n < 2017$, completing the induction. Next, since each a_n is an integer in the range $\{1, 2, \ldots, 2017\}$, it follows that among the first 2018 terms at least one is repeated. That is, there exist indices r and $s > r$ such that $a_r = a_s$. But by the given recurrence, this implies $a_{r+1} = a_{s+1}$, $a_{r+2} = a_{s+2}$, and more generally $a_{r+i} = a_{s+i}$ for all $i \geq 0$. This shows that the sequence $\{a_n\}$ is (ultimately) periodic with period $s - r$.

(b) We will show that $a_{2017} = \boxed{1009}$. From the recurrence we get

$$\begin{aligned} a_n &= \begin{cases} 2a_{n+1} & \text{if } a_n \text{ is even,} \\ 2a_{n+1} - 2017 & \text{if } a_n \text{ is odd} \end{cases} \\ &\equiv 2a_{n+1} \bmod 2017. \end{aligned}$$

Hence

$$a_0 \equiv 2a_1 \equiv 2^2 a_2 \equiv \cdots \equiv 2^{2016} a_{2016} \bmod 2017.$$

Now, by Fermat's Theorem, $2^{2016} \equiv 1 \bmod 2017$ (using the fact that 2017 is a prime number), so $a_{2016} \equiv a_0 = 1 \bmod 2017$. But since the numbers a_n are all positive integers < 2017, this implies $a_{2016} = 1$, and hence $a_{2017} = (1/2)(1 + 2017) = 1009$, as claimed.