2017 UI FRESHMAN MATH CONTEST
September 16, 2017, 10 am – 12 pm

1. Suppose a is a nonzero real number such that $a + \dfrac{1}{a}$ is an integer.

   (a) Prove that $a^2 + \dfrac{1}{a^2}$ is an integer.

   (b) Prove that for any positive integer n, $a^n + \dfrac{1}{a^n}$ is an integer.

2. The number $10! = 3628800$ ends in 2 zeros, $20! = 2432902008176640000$ ends in 4 zeros, and $40! = 8\ldots72000000000$ (a 48 digit number) ends in 9 zeros. How many zeros are at the end of the number $1000!$ (a number with 2568 digits)? Justify your answer.

3. Let a, b, c be real numbers > 1, and let

$$S = \log_a(bc) + \log_b(ca) + \log_c(ab),$$

   where $\log_b x$ denotes the base b logarithm of x. Find, with proof, the smallest possible value of S.

4. Given a positive integer n, let $a_n = \lfloor\sqrt{n}\rfloor$ denote the greatest integer less than or equal to $\sqrt{n}$; for example, $a_5 = \lfloor\sqrt{5}\rfloor = 2$, and $a_{12} = \lfloor\sqrt{12}\rfloor = 3$. How many positive integers $n \le 10^6$ are divisible by a_n? Justify your answer.

5. Let a, b, c, d be positive integers such that $ab = cd$. Show that $a^2+b^2+c^2+d^2$ is a composite number.

6. Define a sequence of positive integers $a_0, a_1, a_2, \ldots$ by $a_0 = 1$ and

$$a_{n+1} = \begin{cases} \dfrac{1}{2}a_n & \text{if } a_n \text{ is even,} \\ \dfrac{1}{2}(a_n + 2017) & \text{if } a_n \text{ is odd,} \end{cases}$$

   for $n = 0, 1, 2, \ldots$.

   (a) Prove that the sequence $\{a_n\}$ is periodic.

   (b) Find, with proof, a_{2017}.