2016 UI FRESHMAN MATH CONTEST
September 24, 2016, 10 am – 12 pm
Solutions

1. Consider the sequence
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots,$$
obtained by writing one 1, two 2's, three 3's, four 4's, etc.

   (a) Find, with proof, the 2016th and 2017th terms in this sequence.

   (b) Find, with proof, a simple general formula for the nth term in the sequence. (The formula can involve the floor or ceiling function.)

Solution. Answers: (a) $\boxed{f(2016) = 63, f(2017) = 64}$; (b) $\boxed{f(n) = \lceil\sqrt{2n + 1/4} - 1/2\rceil}$.
Proofs: (a) The first k blocks occupy $1 + 2 + \cdots + k = k(k + 1)/2$ positions. Thus, we have $f(n) = k$ if and only if n occurs among the last k of these positions, i.e., if and only if

$$\frac{k(k-1)}{2} < n \le \frac{k(k+1)}{2}. \tag{1}$$

For the case $n = 2016$ and 2017, a direct calculation gives the appropriate k-value: $63 \cdot 64/2 = 63 \cdot 32 = 2016$, and $64 \cdot 65/2 = 65 \cdot 32 = 2080$, so the 2016th and 2017th terms of the sequence are, respectively, 63 and 64.
(b) To get a general formula, we multiply (1) by 2 and complete the square on both sides:

$$\left(k - \frac{1}{2}\right)^2 < 2n + \frac{1}{4} \le \left(k + \frac{1}{2}\right)^2,$$
$$k - 1 < \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \le k.$$

This shows that $f(n) = k = \lceil\sqrt{2n + 1/4} - 1/2\rceil$.

2. A unit fraction is a rational number of the form $1/n$, where n is a positive integer.

   (a) Express the number $1/2016$ as a sum of two *distinct* unit fractions.

   (b) Prove that the number $4/2017$ can **not** be expressed as a sum of two *distinct* unit fractions. (You can use the fact that 2017 is a prime number.)

Solution. (a) We have
$$\frac{1}{2016} = \frac{1}{2017} + \frac{1}{2016 \cdot 2017},$$
so $1/2016$ has a representation as a sum of two distinct unit fractions.
(b) Now consider an equation of the form

$$\frac{4}{p} = \frac{1}{x} + \frac{1}{y}, \tag{1}$$

where x and y are distinct positive integers and p is an odd prime. We will show the following:

Claim. If p is a prime satisfying $p \equiv 1 \bmod 4$, then (1) has no solution.

Since 2017 is a prime congruent to 1 mod 4, this proves the assertion of part (b).

Proof of Claim. We argue by contradiction. Suppose $p \equiv 1 \bmod 4$, and that (1) has a solution with distinct positive integers x and y. Without loss of generality, we may assume $x < y$. First note that we must have $x \le p/2$ since if $y > x > p/2$, then $1/x + 1/y < 2/x < 4/p$, contradicting (1).

Clearing denominators in (1), we get

$$4xy = yp + xp = p(x+y). \tag{2}$$

Hence p must divide $4xy$, and since p is an odd prime, it follows that $p|x$ or $p|y$. But since $x < p/2$, p cannot divide x, so we must have $p|y$. Then $y = kp$ for some positive integer k. Dividing (2) by p gives $4kx = kp + x$ and hence

$$x(4k-1) = kp. \tag{3}$$

Hence k divides $x(4k-1)$, and since k is coprime with $4k-1$, it must divide x. Thus $x = mk$ for some positive integer m. Substituting this into (3) we get $p = m(4k-1)$, and since p is prime, this is only possible if $m = 1$ and $p = 4k-1$. But this contradicts our assumption $p \equiv 1 \bmod 4$, thereby proving the claim.

3. Let N be a positive integer. Prove that there exist positive integers m and n such that

$$\frac{1}{N} < 2016\sqrt{m} - 2017\sqrt{n} < \frac{2}{N},$$

and determine suitable choices for m and n explicitly.

Solution. (Variation of B1, Putnam 2011)

Answers: $\boxed{n = 2016^2N^2, m = 2017^2N^2 + 4}$ is a pair of integers with the desired property.

Proof: First note that, for any positive real numbers x and y with $y \le x$,

$$\sqrt{x+y} - \sqrt{x} = \int_x^{x+y} \frac{1}{2\sqrt{t}}dt \begin{cases} < \dfrac{y}{2\sqrt{x}}, \\ \ge \dfrac{y}{2\sqrt{x+y}} > \dfrac{y}{2\sqrt{2x}} > \dfrac{y}{2\sqrt{2}\sqrt{x}}. \end{cases} \tag{1}$$

Now let $n = 2016^2N^2$ and $m = 2017^2N^2 + h$, where h is a positive integer to be determined. Then, using (1) with $x = (2016 \cdot 2017 \cdot N)^2$, $y = 2016^2h$, we get

$$2016\sqrt{m} - 2017\sqrt{n} = \sqrt{2016^2 2017^2 N^2 + 2016^2 h} - \sqrt{2016^2 2016^2 N^2} \begin{cases} < \dfrac{2016^2 h}{2 \cdot 2016 \cdot 2017 \cdot N} < \dfrac{h}{2N}, \\ > \dfrac{2016^2 h}{2\sqrt{2} \cdot 2016 \cdot 2017 \cdot N} > \dfrac{h}{4N}, \end{cases}$$

provided h satisfies $h \le 2017^2N^2$. Setting $h = 4$, the above bounds become $2/N$ and $1/N$, respectively, and we obtain the desired inequalities.

Alternative solution. Here is a shorter, alternative approach to proving the *existence* of n and m satisfying the given inequality.

Consider the sequences $x_n = 2016\sqrt{n}$ and $y_n = 2017\sqrt{n}$. These sequences are strictly increasing sequences of real numbers with limit ∞. Moreover, the difference between consecutive terms in these sequences tends to 0 since $\sqrt{n+1} - \sqrt{n} \to 0$. Hence, if K is sufficiently large, then the interval $(K, K+0.5/N)$ contains a number, y_n, from the second sequence and the interval $(K+1.5/N, K+2/N)$ contains a number, x_m, from the first sequence. Then $1/N < x_m - y_n < 2/N$, so the numbers n and m chosen in this way have the desired property.

Remark. This is a variation of Problem B1 in the 2011 Putnam Exam, which involved an inequality of the form $\epsilon < h\sqrt{n} - k\sqrt{m} < 2\epsilon$, where h and k are general positive integers and ϵ is any positive real number. For more details see the solutions posted at Kiran Kedlaya's Putnam Archive, `http://kskedlaya.org/putnam-archive/`.

4. Suppose $a_1, b_1, \ldots, a_n, b_n$ are positive real numbers satisfying $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. Prove that

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \geq \frac{1}{2}.$$

Solution. Note first that

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} - \sum_{i=1}^n \frac{b_i^2}{a_i + b_i} = \sum_{i=1}^n \frac{a_i^2 - b_i^2}{a_i + b_i} = \sum_{i=1}^n (a_i - b_i) = 0,$$

so we have

$$(1) \qquad \sum_{i=1}^n \frac{a_i^2}{a_i + b_i} = \sum_{i=1}^n \frac{b_i^2}{a_i + b_i}, = \frac{1}{2} \sum_{i=1}^n \frac{a_i^2 + b_i^2}{a_i + b_i}.$$

Also, since $(a+b)^2 \leq (a^2 + b^2)(1^2 + 1^2) = 2(a^2 + b^2)$ by Cauchy's inequality, we have

$$(2) \qquad \frac{a_i^2 + b_i^2}{a_i + b_i} \geq \frac{(a_i + b_i)^2/2}{a_i + b_i} = \frac{a_i + b_i}{2}.$$

From (1) and (2) we get

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} = \frac{1}{2} \sum_{i=1}^n \frac{a_i^2 + b_i^2}{a_i + b_i} \geq \frac{1}{2} \sum_{i=1}^n \frac{a_i + b_i}{2} = \frac{1}{2},$$

which proves the desired bound.

Alternative solution. Using Cauchy's inequality, we have

$$\begin{aligned} \left(\sum_{i=1}^n a_i \right)^2 &= \left(\sum_{i=1}^n \frac{a_i}{\sqrt{a_i + b_i}} \cdot \sqrt{a_i + b_i} \right)^2 \\ &\leq \left(\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \right) \left(\sum_{i=1}^n (a_i + b_i) \right) \\ &= \left(\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \right) \cdot 2 \end{aligned}$$

and hence

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \geq \frac{1}{2} \left(\sum_{i=1}^n a_i \right)^2 = \frac{1}{2}.$$

5. Let $x_0 = 0$, $x_1 = 1$, and for $n \geq 1$, let

$$x_{n+1} = \frac{1}{n+1} x_n + \left(1 - \frac{1}{n+1} \right) x_{n-1}.$$

Show that the sequence $\{x_n\}$ converges as $n \to \infty$ and determine its limit.

Solution. We will show that

$$(1) \qquad \lim_{n \to \infty} x_n = \boxed{\ln 2}.$$

For the proof, write the recurrence in the form

$$x_{n+1}(n+1) = x_n + n x_{n-1} \quad (n \geq 1).$$

Setting $d_n = x_{n+1} - x_n$ and simplifying, we deduce $d_n = (-n/(n+1))d_{n-1}$ for $n \geq 1$. Iterating this relation, we get

$$d_n = \frac{-n}{n+1} \cdot \frac{-(n-1)}{n} \cdots \frac{-1}{2} d_0 = \frac{(-1)^n}{n+1},$$

since $d_0 = x_1 - x_0 = 1$. Hence

$$x_n = x_0 + \sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} \frac{(-1)^i}{i+1}.$$

The last series is an alternating series with decreasing terms and thus converges, with sum

$$\sum_{i=0}^{\infty} (-1)^i \frac{1}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2,$$

by the Taylor series for $\ln(1+x)$.

6. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Find, with proof, $f(100)$.

Solution. Answer: $f(100) = \boxed{181}$.

Proof. First consider the value $f(1)$. Since f maps into the positive integers, we must have either (i) $f(1) = 1$, (ii) $f(1) = 2$, or $f(1) = m$ for some integer $m \geq 3$.

In case (i), the second property gives $3 = f(f(1)) = f(1) = 1$, which is a contradiction. In case (iii), the first property yields $f(m) > f(m-1) > \cdots > f(1)$, so $f(m) \geq f(1) + (m-1) = 2m - 1 \geq 5$, while the second property yields $f(m) = f(f(1)) = 3 < 5$, which is again a contradiction. Thus, we must have $f(1) = 2$.

Now, starting with $f(1) = 2$, and using the relation $f(f(n)) = 3n$. we get the following values:

$f(2) = 3, f(3) = 6, f(6) = 9, f(9) = 18, f(18) = 27, f(27) = 54, f(54) = 81, f(81) = 162$.

Also, using the relation $f(n+1) > f(n)$ and the already computed values $f(3) = 6$, $f(9) = 9$ we see that for $3 \leq n \leq 6$, $f(n)$ must take on the values $6, 7, 8, 9$ in order. Hence, we must have $f(4) = 7$, and the relation $f(f(n)) = 3n$ then gives:

$f(7) = 12, f(12) = 21, f(21) = 36, f(36) = 63, f(63) = 108, f(108) = 189$.

Now, since $f(81) = 162$ and $f(108) = 189$, and there are exactly 27 integers in each of the intervals $[81, 108]$ and $[162, 189]$, we must have $f(81+n) = 162 + n$ for $n = 1, \ldots, 27$, and in particular, $f(100) = 162 + 19 = \boxed{181}$.

Remark. More generally, we have, for any integers $k \geq 0$ and $0 \leq m < 3^k$, the explicit formula

$$f(3^k + m) = 2 \cdot 3^k + m. \tag{1}$$

For the proof of (1), start as before by showing that $f(1) = 2$, and $f(2) = 3$, then use induction to show that, for any nonnegative integer k,

$$f(3^k) = 2 \cdot 3^k, \quad f(2 \cdot 3^k) = 3^{k+1}. \tag{2}$$

From (2) and the property $f(n+1) > f(n)$ it follows that the 3^k+1 values $f(3^k), f(3^k+1), \ldots, f(3^k + 3^k$, form an increasing sequence of 3^k distinct integers, ranging from $2 \cdot 3^k$ to 3^{k+1}. Since there are exactly $3^k + 1$ integers in the interval $[2 \cdot 3^k, 3^{k+1}]$, these values must fill the entire interval, i.e., we have $f(3^k + m) = 3^k + m$ for $0 \leq m < 3^k$. This proves (1).