2016 UI FRESHMAN MATH CONTEST
September 24, 2016, 10 am – 12 pm

1. Consider the sequence
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots,$$
obtained by writing one 1, two 2's, three 3's, four 4's, etc.
   (a) Find, with proof, the 2016th and 2017th terms in this sequence.
   (b) Find, with proof, a simple general formula for the nth term in the sequence. (The formula can involve the floor or ceiling function.)

2. A unit fraction is a rational number of the form $1/n$, where n is a positive integer.
   (a) Express the number $1/2016$ as a sum of two *distinct* unit fractions.
   (b) Prove that the number $4/2017$ can **not** be expressed as a sum of two *distinct* unit fractions. (You can use the fact that 2017 is a prime number.)

3. Let N be a positive integer. Prove that there exist positive integers m and n such that
$$\frac{1}{N} < 2016\sqrt{m} - 2017\sqrt{n} < \frac{2}{N},$$
and determine suitable choices for m and n explicitly.

4. Suppose $a_1, b_1, \dots, a_n, b_n$ are positive real numbers satisfying $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. Prove that
$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \geq \frac{1}{2}.$$

5. Let $x_0 = 0$, $x_1 = 1$, and for $n \geq 1$, let
$$x_{n+1} = \frac{1}{n+1} x_n + \left(1 - \frac{1}{n+1}\right) x_{n-1}.$$
Show that the sequence $\{x_n\}$ converges as $n \to \infty$ and determine its limit.

6. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Find, with proof, $f(100)$.

[Solutions at `http://www.math.uiuc.edu/contests.html`]