2015 UI FRESHMAN MATH CONTEST
September 26, 2015, 10 am – 12 pm
Solutions

1. Let N be the number

$$N = 12345678910111213 14 \dots 99100$$

obtained by writing the integers $1, 2, 3, 4 \dots, 99, 100$ next to each other. What is the remainder of N when divided by 9? Explain!

Solution. We will show that the remainder is $\boxed{1}$.
By the divisibility test for 9, we have $N \equiv s(N) \bmod 9$, where $s(N)$ the sum of digits of N modulo 9. Now,

$$s(N) = \sum_{k=1}^{100} s(k) \equiv \sum_{k=1}^{100} k = 50 \cdot 101 \equiv 5 \cdot 2 \equiv 1 \bmod 9,$$

so $N \equiv 1 \bmod 9$.

2. Prove that the equation

$$x^2 + y^2 + z^2 = 2xyz$$

has no solution in positive integers x, y, z.

Solution. We argue by contradiction. Assume the equation $x^2 + y^2 + z^2 = 2xyz$ has a solution in positive integers x, y, z. Let 2^k be the highest power of 2 dividing all of x, y, z, and set $x_1 = x/2^k$, $y_1 = y/2^k$, $z_1 = z/2^k$. Then at least one of x_1, y_1, z_1 is odd. Dividing the given equation by $(2^k)^2$, we get

$$(*) \qquad x_1^2 + y_1^2 + z_1^2 = 2 \cdot 2^k x_1 y_1 z_1.$$

Now consider congruences modulo 4 in $(*)$. Since, for any integer n, $n^2 \equiv 0 \bmod 4$ if n is even, and $n^2 \equiv 1 \bmod 4$ if n is odd, we have for the left side of $(*)$:

(1) $x_1^2 + y_1^2 + z_1^2 \equiv 1 \bmod 4$ if one of the integers x_1, y_1, z_1 is odd and two are even;
(2) $x_1^2 + y_1^2 + z_1^2 \equiv 2 \bmod 4$ if two of the integers x_1, y_1, z_1 are odd and one is even;
(3) $x_1^2 + y_1^2 + z_1^2 \equiv 3 \bmod 4$ if all three of the integers x_1, y_1, z_1 are odd.

On the other hand, in cases (1) and (2) the right side of $(*)$ is divisible by 4, hence congruent to 0 modulo 4, and in case (3) the right side is divisible by (at least) 2 and hence congruent to 0 or 2 modulo 4. Thus, in either case we have a contradiction, and the proof is complete.

3. Let x, y, z be arbitrary real numbers in the interval $[0, 1]$, and let $u = x(1 - y)$, $v = y(1 - z)$, $w = z(1 - x)$. Prove that at least one of the numbers u, v, w is $\leq 1/4$.

Solution. Consider the product $(*)$ $uvw = (x(1-x))(y(1-y))(z(1-z))$. Note that the function $f(t) = t(1-t)$ satisfies $f'(t) > 0$ if $t < 1/2$, $f'(t) < 0$ if $t > 1/2$, and thus has a unique maximum at $t = 1/2$, with value $f(1/2) = 1/4$. Hence each of the three factors on the right-hand side of $(*)$ is $\leq 1/4$. Therefore at least one of the three factors u, v, w on the left must be $\leq 1/4$. This is what we had to prove.

4. Prove that, given any 9 pairwise distinct lattice points, $P_1, \dots, P_9$, in 3-dimensional space, there exist two of these points, say P_i and P_j with $i \neq j$, such that the line segment P_iP_j contains another lattice point (different from P_i and P_j). (A lattice point is a point with integer coordinates.)

Solution. We use a parity argument to show that there exist i and j with $i \neq j$ such that the midpoint of the line segment P_iP_j is a lattice point.

First note that there are $2^3 = 8$ possible combinations for the parities of the coordinates of a 3-dimensional lattice point: $(even, even, even), (even, even, odd), \ldots, (odd, odd, odd)$. Thus, among the 9 given lattice points there must be two, say P_i and P_j, whose coordinates have matching parities. It follows that the vector P_iP_j (i.e., the vector whose components are the differences of the components of P_i and P_j) has all even coordinates. Hence the vector $(1/2)P_iP_j$, which represents the midpoint of the line segment from P_i to P_j, has all integer coordinates. This proves our claim.

5. Given a point $P_0 = (x_0, y_0, z_0)$ in 3-dimensional space, define a sequence of points $P_k = (x_k, y_k, z_k)$ by

$$\begin{aligned} x_{k+1} &= x_k - y_k, \\ y_{k+1} &= y_k - z_k, \\ z_{k+1} &= z_k - x_k, \end{aligned}$$

for $k = 0, 1, 2, \ldots$ For example, if $P_0 = (5, 3, 4)$, then $P_1 = (5-3, 3-4, 4-5) = (2, -1, -1)$, $P_2 = (3, 0, -3)$, $P_3 = (3, 3, -6)$, etc.

Prove that, if the coordinates x_0, y_0, z_0 of the initial point P_0 are not all equal, then $|P_k| \to \infty$ as $k \to \infty$, where $|P_k|$ denotes the distance of P_k to the origin.

Solution. First note that, by the given recurrence,

$$x_{k+1} + y_{k+1} + z_{k+1} = (x_k - y_k) + (y_k - z_k) + (z_k - x_k) = 0.$$

so $x_k + y_k + z_k = 0$ for all $k \geq 1$.

Now, for any $k \geq 1$,

$$\begin{aligned} |P_{k+1}|^2 &= x_{k+1}^2 + y_{k+1}^2 + z_{k+1}^2 \\ &= (x_k - y_k)^2 + (y_k - z_k)^2 + (z_k - x_k)^2 \\ &= 2(x_k^2 + y_k^2 + z_k^2) - 2(x_k y_k + y_k z_k + z_k x_k) \\ &= 3(x_k^2 + y_k^2 + z_k^2) - (x_k + y_k + z_k)^2 \\ &= 3|P_k|^2, \end{aligned}$$

since $(x_k + y_k + z_k)^2 = 0$ by the above observation. Hence $|P_{k+1}| = \sqrt{3}|P_k|$ for all $k \geq 1$, and iterating this relation we get $(*)$ $|P_{k+1}| = 3^{k/2}|P_1|$. The assumption that the coordinates of P_0 are not all equal guarantees that $P_1 = (x_0 - y_0, y_0 - z_0, z_0 - x_0)$ has at least one non-zero coordinate, so $|P_1| > 0$, and by $(*)$ it follows that $|P_k| \to \infty$ as $k \to \infty$.

6. Let a, b be arbitrary real numbers, and let the sequence x_n be defined by $x_0 = a, x_1 = b$, and

$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2}) \quad (n = 2, 3, \ldots).$$

Prove that as $n \to \infty$, x_n converges to a limit, and find a formula for this limit in terms of a and b.

Solution. We will show that $(*)$ $\lim_{n\to\infty} x_n = (1/3)a + (2/3)b$. For the proof, rewrite the given recurrence as

$$x_n - x_{n-1} = -\frac{1}{2}(x_{n-1} - x_{n-2}).$$

Iterating this relation, we get

$$x_n - x_{n-1} = \left(-\frac{1}{2}\right)^{n-1}(x_1 - x_0) = \left(-\frac{1}{2}\right)^{n-1}(b - a)$$

for $n \geq 1$. It follows that

$$x_n = x_0 + \sum_{k=1}^{n}(x_k - x_{k-1}) = a + (b-a)\sum_{k=1}^{n}(-1/2)^{k-1} = a + (b-a)\frac{1-(-1/2)^n}{1-(-1/2)}$$

As $n \to \infty$, the right side converges, with limit $a + (b-a)(2/3) = (1/3)a + (2/3)b$. This proves $(*)$.