1. Let \( N \) be the number

\[
N = 1234567891011121314\ldots99100
\]

obtained by writing the integers 1, 2, 3, 4, \ldots, 99, 100 next to each other. What is the remainder of \( N \) when divided by 9? Explain!

**Solution.** We will show that the remainder is 1.

By the divisibility test for 9, we have \( N \equiv s(N) \mod 9 \), where \( s(N) \) the sum of digits of \( N \) modulo 9. Now,

\[
s(N) = \sum_{k=1}^{100} s(k) \equiv \sum_{k=1}^{100} k = 50 \cdot 101 \equiv 5 \cdot 2 \equiv 1 \mod 9,
\]

so \( N \equiv 1 \mod 9 \).

2. Prove that the equation

\[ x^2 + y^2 + z^2 = 2xyz \]

has no solution in positive integers \( x, y, z \).

**Solution.** We argue by contradiction. Assume the equation \( x^2 + y^2 + z^2 = 2xyz \) has a solution in positive integers \( x, y, z \). Let \( 2^k \) be the highest power of 2 dividing all of \( x, y, z \), and set \( x_1 = x/2^k \), \( y_1 = y/2^k \), \( z_1 = z/2^k \).

Then at least one of \( x_1, y_1, z_1 \) is odd. Dividing the given equation by \( (2^k)^2 \), we get

\[ x_1^2 + y_1^2 + z_1^2 = 2 \cdot 2^k x_1 y_1 z_1. \]  

(\*) Now consider congruences modulo 4 in (\*). Since, for any integer \( n \), \( n^2 \equiv 0 \mod 4 \) if \( n \) is even, and \( n^2 \equiv 1 \mod 4 \) if \( n \) is odd, we have for the left side of (\*):

1. \( x_1^2 + y_1^2 + z_1^2 \equiv 1 \mod 4 \) if one of the integers \( x_1, y_1, z_1 \) is odd and two are even;
2. \( x_1^2 + y_1^2 + z_1^2 \equiv 2 \mod 4 \) if two of the integers \( x_1, y_1, z_1 \) are odd and one is even;
3. \( x_1^2 + y_1^2 + z_1^2 \equiv 3 \mod 4 \) if all three of the integers \( x_1, y_1, z_1 \) are odd.

On the other hand, in cases (1) and (2) the right side of (\*) is divisible by 4, hence congruent to 0 modulo 4, and in case (3) the right side is divisible by (at least) 2 and hence congruent to 0 or 2 modulo 4. Thus, in either case we have a contradiction, and the proof is complete.

3. Let \( x, y, z \) be arbitrary real numbers in the interval \([0, 1]\), and let \( u = x(1 - y), v = y(1 - z), w = z(1 - x) \). Prove that at least one of the numbers \( u, v, w \) is \( \leq 1/4 \).

**Solution.** Consider the product (\*) \( uvw = (x(1-x))(y(1-y))(z(1-z)) \). Note that the function \( f(t) = t(1-t) \) satisfies \( f'(t) > 0 \) if \( t < 1/2 \), \( f'(t) < 0 \) if \( t > 1/2 \), and thus has a unique maximum at \( t = 1/2 \), with value \( f(1/2) = 1/4 \). Hence each of the three factors on the right-hand side of (\*) is \( \leq 1/4 \). Therefore at least one of the three factors \( u, v, w \) on the left must be \( \leq 1/4 \). This is what we had to prove.

4. Prove that, given any 9 pairwise distinct lattice points, \( P_1, \ldots, P_9 \), in 3-dimensional space, there exist two of these points, say \( P_i \) and \( P_j \) with \( i \neq j \), such that the line segment \( P_iP_j \) contains another lattice point (different from \( P_i \) and \( P_j \)). (A lattice point is a point with integer coordinates.)
6. Let \( P \) be an arbitrary real number, and let the sequence \( x_n \) be defined by \( x_0 = a, x_1 = b, \) and
\[
x_n = \frac{1}{2} \left( x_{n-1} + x_{n-2} \right) \quad (n = 2, 3, \ldots).
\]
Prove that as \( n \to \infty \), \( x_n \) converges to a limit, and find a formula for this limit in terms of \( a \) and \( b \).

**Solution.** We will show that \((*)\) \( \lim_{n \to \infty} x_n = (1/3)a + (2/3)b \). For the proof, rewrite the given recurrence as
\[
x_n - x_{n-1} = -\frac{1}{2} (x_{n-1} - x_{n-2}).
\]
Iterating this relation, we get
\[
x_n - x_{n-1} = \left( -\frac{1}{2} \right)^{n-1} (x_1 - x_0) = \left( -\frac{1}{2} \right)^{n-1} (b - a)
\]
for \( n \geq 1 \). It follows that
\[
x_n = x_0 + \sum_{k=1}^{n} (x_k - x_{k-1}) = a + (b - a) \sum_{k=1}^{n} (-1/2)^{k-1} = a + (b - a) \frac{1 - (-1/2)^n}{1 - (-1/2)}
\]
As \( n \to \infty \), the right side converges, with limit \( a + (b - a)(2/3) = (1/3)a + (2/3)b \). This proves \((*)\).