# 2015 UI FRESHMAN MATH CONTEST
# September 26, 2015, 10 am – 12 pm

1. Let $N$ be the number
$$N = 1234567891011121314\ldots 99100$$
obtained by writing the integers $1, 2, 3, 4\ldots, 99, 100$ next to each other. What is the remainder of $N$ when divided by 9? Explain!

2. Prove that the equation
$$x^2 + y^2 + z^2 = 2xyz$$
has no solution in positive integers $x, y, z$.

3. Let $x, y, z$ be arbitrary real numbers in the interval $[0, 1]$, and let $u = x(1 - y)$, $v = y(1 - z)$, $w = z(1 - x)$. Prove that at least one of the numbers $u, v, w$ is $\le 1/4$.

4. Prove that, given any 9 pairwise distinct lattice points, $P_1, \ldots, P_9$, in 3-dimensional space, there exist two of these points, say $P_i$ and $P_j$ with $i \neq j$, such that the line segment $P_iP_j$ contains another lattice point (different from $P_i$ and $P_j$). (A lattice point is a point with integer coordinates.)

5. Given a point $P_0 = (x_0, y_0, z_0)$ in 3-dimensional space, define a sequence of points $P_k = (x_k, y_k, z_k)$ by
$$x_{k+1} = x_k - y_k,$$
$$y_{k+1} = y_k - z_k,$$
$$z_{k+1} = z_k - x_k,$$
for $k = 0, 1, 2, \ldots$ For example, if $P_0 = (5, 3, 4)$, then $P_1 = (5 - 3, 3 - 4, 4 - 5) = (2, -1, -1)$, $P_2 = (3, 0, -3)$, $P_3 = (3, 3, -6)$, etc.

   Prove that, if the coordinates $x_0, y_0, z_0$ of the initial point $P_0$ are not all equal, then $|P_k| \to \infty$ as $k \to \infty$, where $|P_k|$ denotes the distance of $P_k$ to the origin.

6. Let $a, b$ be arbitrary real numbers, and let the sequence $x_n$ be defined by $x_0 = a, x_1 = b$, and
$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2}) \quad (n = 2, 3, \ldots).$$
Prove that as $n \to \infty$, $x_n$ converges to a limit, and find a formula for this limit in terms of $a$ and $b$.

[Solutions at http://www.math.uiuc.edu/contests.html]