2014 UI FRESHMAN MATH CONTEST
September 27, 2014, 10 am – 12 pm
Solutions

1. (AIME 1988) For any positive integer k, let $f(k)$ denote the sum of the squares of the digits of k (expressed in decimal), and let f_n denote the n-th iterate of f, defined by $f_1(k) = f(k)$ and $f_n(k) = f(f_{n-1}(k))$ for $n = 2, 3, \dots$. (For example, $f_1(2014) = f(2014) = 2^2 + 0^2 + 1^2 + 4^2 = 21$, $f_2(2014) = f(f_1(2014) = f(21) = 2^2 + 1^1 = 5$.)

   Find, with proof, $f_{2014}(11)$.

   Solution. Answer: $\boxed{89}$. Calculating the first few terms of the sequence $f_1(11), f_2(11), \dots$ gives $2, 4, 16, 37, 58, 89, 145, 42, 20, 4, \dots$. Thus, from the second term onwards, the sequence is periodic with period 8. Since $2014 \equiv 6 \bmod 8$, we get $f_{2014}(11) = f_6(11) = 89$.

2. (A1, Putnam 2003) For any positive integer n, let $f(n)$ be the number of ways to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$. For example, with $n = 4$ there are four such representations, $4 = 4, 4 = 2 + 2, 4 = 1 + 1 + 2, 4 = 1 + 1 + 1 + 1$, so $f(4) = 4$.

   Find, with proof, a general formula for $f(n)$.

   Solution. We will show that $\boxed{f(n) = n}$ for all $n \in \mathbb{N}$. A representation of n of the desired form can be written as

   $$n = qa_1 + (k - q)(a_1 + 1) = ka_1 + (k - q), \qquad (1)$$

   where q is the number of terms a_1 in the representation. Clearly, $1 \le q \le k$, so $0 \le k - q \le k - 1$. The latter constraint shows that (1) is exactly the representation obtained from the division algorithm upon dividing n by k with remainder $k - q$ and quotient a_1. It follows that for each $k = 1, \dots, n$ there exists exactly one such representation, so the total number of representations of the desired form is n, as claimed.

3. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying $0 \le x_i \le 1$ for each i. Prove that

   $$(1 + x_1)(1 + x_2) \dots (1 + x_n) \le 2^{n-1}(1 + x_1 x_2 \dots x_n).$$

   Solution. We use induction on n. In the base case, $n = 1$, the inequality reduces to $(1 + x_1) \le 1 + x_1$, which holds trivially.
   Now let $k \ge 1$ be given and suppose the inequality holds for $n = k$. Set

   $$P_k = \prod_{i=1}^{k}(1 + x_i), \quad Q_k = \prod_{i=1}^{k} x_i.$$

   Then we have

   $$\begin{aligned} P_{k+1} &= P_k(1 + x_{k+1}) \\ &\le 2^{k-1}(1 + Q_k)(1 + x_{k+1}) \quad \text{(by induction hypothesis))} \\ &= 2^{k-1}(1 + Q_k x_{k+1} + Q_k + x_{k+1}) \\ &= 2^{k-1}(2 + 2Q_k x_{k+1}) \quad \text{(since } (*)\ a + b \le 1 + ab \text{ for } 0 \le a, b \le 1) \\ &= 2^k(1 + Q_{k+1}), \end{aligned}$$

   which proves the inequality for $n = k + 1$ and completes the induction. (The inequality $(*)$ follows from $0 \le (1 - a)(1 - b) = 1 + ab - a - b$.)

4. The harmonic mean of two numbers x and y is defined as $H(x,y) = 2/(1/x + 1/y)$; for example, $H(2,6) = 2/(1/2 + 1/6) = 3$.

   Prove that any *odd* prime number $p \geq 3$ can be expressed as the harmonic mean of a *unique* pair of positive integers (x, y) with $x < y$.

   Solution. We will show that $\boxed{x = (p+1)/2,\ y = p(p+1)/2}$ is the unique positive integer solution to the equation $p = H(x,y)$ satisfying $x < y$.
   A direct calculation shows that this pair satisfies

   $$\frac{1}{x} + \frac{1}{y} = \frac{2}{p+1} + \frac{2}{p(p+1)} = \frac{2}{p}.$$

   and hence $H(x,y) = 2/(1/x + 1/y) = 2/(2/p) = p$, showing that (x,y) is indeed a solution.
   To prove uniqueness, first note that any such pair (x,y) must satisfy (1) $x < p < y$: If $y > x \geq p$, then $H(x,y) > 2/(2/p) = p$, and similarly, if $x < y \leq p$, then $H(x,y) < p$.
   Now suppose that $p = H(x,y)$ for some positive integers x and y with $x < p < y$. Clearing denominators, we can write the equation $p = H(x,y)$ as $p = 2xy/(x+y)$ or equivalently (2) $p(x+y) = 2xy$. The latter form shows that p must divide $2xy$, and since p is an odd prime and $p > x$ (by (1)), p must divide y.
   Hence $y = kp$ for some positive integer k. Substituting this into (2) we get $p(x + kp) = 2kpx$, or (3) $kp = x(2k-1)$. Thus k must divide $x(2k-1)$, and since k is coprime with $2k-1$, it must divide x. It follows that $x = mk$ for some positive integer m. Substituting this into (3) we get $p = m(2k-1)$, and since p is prime, we must have either (i) $m = 1$ and $k = (p+1)/2$ or (ii) $m = p$ and $k = 1$.
   In case (ii) we would have $x = mk = p \cdot 1 = p$ contradicting the condition $x < p$. This leaves case (i) as the only solution. In this case, we get $x = mk = (p+1)/2$ and $y = kp = p(p+1)/2$, which is the pair definned above. Thus, this pair is indeed the only solution.

5. (Variation of A3, Putnam 2009) Evaluate the determinant

   $$\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

   (The argument of cos is in radians, not degrees.)

   Solution. We will show that the determinant is $\boxed{0}$.
   Adding the third column to the first column does not affect the determinant, but changes the entries in the first column to $\cos 1 + \cos 3, \cos 4 + \cos 6, \cos 7 + \cos 9$, respectively. Now, using the identity

   $$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

   we get

   $$\cos 3 + \cos 1 = 2\cos 2\cos 1,$$
   $$\cos 6 + \cos 4 = 2\cos 5\cos 1,$$
   $$\cos 9 + \cos 7 = 2\cos 8\cos 1,$$

   Thus, the new first column is a scalar multiple of the second column. Hence the matrix has determinant 0.

6. (B3, Putnam 1993) For any real number t, let $\langle t \rangle$ denote the integer closest to t; for example, $\langle 3.14159 \rangle = 3$ and $\langle 2.71828 \rangle = 3$.

   If x and y are chosen at random in the interval $(0,1)$, what is the probability that the number $\langle x/y \rangle$ is even? Justify your reasoning, and express your answer in the form $r + sc$, where r and s are rational numbers and c is a famous constant.

Solution. Answer: $\boxed{5/4 - \pi/4}$ For each $n = 0, 1, 2, \ldots$ let p_n denote the probability that $\langle x/y \rangle = 2n$. Then the probability that x/y is even is equal to $\sum_{n=0}^{\infty} p_n$.
For $n = 0$ we have $\langle x/y \rangle = 0$ if and only if $x/y < 1/2$, i.e., if and only if $y > 2x$. The latter condition represents a triangle with vertices $(0,0), (1/2, 1), (0, 1)$ inside the unit square $[0,1] \times [0,1]$, and the probability p_0 is equal to the area of this triangle, i.e., $p_0 = 1/4$.
For $n = 1, 2, \ldots$ we have

$$\begin{aligned}\langle x/y \rangle = 2n &\iff 2n - 1/2 \le \frac{x}{y} \le 2n + 1/2 \\ &\iff (4n-1)y \le x \le (4n+1)y \\ &\iff \frac{1}{4n+1}x \le y \le \frac{1}{4n-1}x.\end{aligned}$$

The latter inequalities represent a triangular region inside the unit square $[0,1] \times [0,1]$, and p_n is the area of this region. Calculating this area we get

$$p_n = \frac{1}{2}\left(\frac{1}{4n-1} - \frac{1}{4n+1}\right).$$

Thus, the probability we seek is

$$\begin{aligned}\sum_{n=0}^{\infty} p_n &= \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{4n-1} - \frac{1}{4n+1}\right) \\ &= \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots = \frac{1}{4} - \left(\frac{\pi}{4} - 1\right)\end{aligned}$$

using Leibnitz' formula for $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$.