1. (AIME 1988) For any positive integer \( k \), let \( f(k) \) denote the sum of the squares of the digits of \( k \) (expressed in decimal), and let \( f_n \) denote the \( n \)-th iterate of \( f \), defined by \( f_1(k) = f(k) \) and \( f_n(k) = f(f_{n-1}(k)) \) for \( n = 2, 3, \ldots \). (For example, \( f_1(2014) = f(2014) = 2^2 + 0^2 + 1^2 + 4^2 = 21 \), \( f_2(2014) = f(f_1(2014) = f(21) = 2^2 + 1^1 = 5 \).) Find, with proof, \( f_{2014}(11) \).

2. (A1, Putnam 2003) For any positive integer \( n \), let \( f(n) \) be the number of ways to write \( n \) as a sum of positive integers, \( n = a_1 + a_2 + \cdots + a_k \), with \( k \) an arbitrary positive integer and \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1 \). For example, with \( n = 4 \) there are four such representations, \( 4 = 4 \), \( 4 = 2 + 2 \), \( 4 = 1 + 1 + 2 \), \( 4 = 1 + 1 + 1 + 1 \), so \( f(4) = 4 \). Find, with proof, a general formula for \( f(n) \).

3. Let \( x_1, x_2, \ldots, x_n \) be real numbers satisfying \( 0 \leq x_i \leq 1 \) for each \( i \). Prove that
\[
(1 + x_1)(1 + x_2)\ldots(1 + x_n) \leq 2^{n-1}(1 + x_1 x_2 \ldots x_n).
\]

4. The harmonic mean of two numbers \( x \) and \( y \) is defined as \( H(x, y) = 2/(1/x + 1/y) \); for example, \( H(2, 6) = 2/(1/2 + 1/6) = 3 \). Prove that any odd prime number \( p \geq 3 \) can be expressed as the harmonic mean of a unique pair of positive integers \( (x, y) \) with \( x < y \).

5. (Variation of A3, Putnam 2009) Evaluate the determinant
\[
\begin{vmatrix}
\cos 1 & \cos 2 & \cos 3 \\
\cos 4 & \cos 5 & \cos 6 \\
\cos 7 & \cos 8 & \cos 9 \\
\end{vmatrix}
\]
(The argument of \( \cos \) is in radians, not degrees.)

6. (B3, Putnam 1993) For any real number \( t \), let \( \langle t \rangle \) denote the integer closest to \( t \); for example, \( \langle 3.14159 \rangle = 3 \) and \( \langle 2.71828 \rangle = 3 \). If \( x \) and \( y \) are chosen at random in the interval \((0, 1)\), what is the probability that the number \( \langle x/y \rangle \) is even? Justify your reasoning, and express your answer in the form \( r + sc \), where \( r \) and \( s \) are rational numbers and \( c \) is a famous constant.

[Solutions at http://www.math.uiuc.edu/contests.html]