# 2014 UI FRESHMAN MATH CONTEST
# September 27, 2014, 10 am – 12 pm

1. (AIME 1988) For any positive integer $k$, let $f(k)$ denote the sum of the squares of the digits of $k$ (expressed in decimal), and let $f_n$ denote the $n$-th iterate of $f$, defined by $f_1(k) = f(k)$ and $f_n(k) = f(f_{n-1}(k))$ for $n = 2, 3, \dots$. (For example, $f_1(2014) = f(2014) = 2^2 + 0^2 + 1^2 + 4^2 = 21$, $f_2(2014) = f(f_1(2014) = f(21) = 2^2 + 1^1 = 5$.)

   Find, with proof, $f_{2014}(11)$.

2. (A1, Putnam 2003) For any positive integer $n$, let $f(n)$ be the number of ways to write $n$ as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with $k$ an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$. For example, with $n = 4$ there are four such representations, $4 = 4, 4 = 2 + 2, 4 = 1 + 1 + 2, 4 = 1 + 1 + 1 + 1$, so $f(4) = 4$.

   Find, with proof, a general formula for $f(n)$.

3. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying $0 \le x_i \le 1$ for each $i$. Prove that

$$(1 + x_1)(1 + x_2) \dots (1 + x_n) \le 2^{n-1}(1 + x_1 x_2 \dots x_n).$$

4. The harmonic mean of two numbers $x$ and $y$ is defined as $H(x, y) = 2/(1/x + 1/y)$; for example, $H(2, 6) = 2/(1/2 + 1/6) = 3$.

   Prove that any *odd* prime number $p \ge 3$ can be expressed as the harmonic mean of a *unique* pair of positive integers $(x, y)$ with $x < y$.

5. (Variation of A3, Putnam 2009) Evaluate the determinant

$$\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

   (The argument of cos is in radians, not degrees.)

6. (B3, Putnam 1993) For any real number $t$, let $\langle t \rangle$ denote the integer closest to $t$; for example, $\langle 3.14159 \rangle = 3$ and $\langle 2.71828 \rangle = 3$.

   If $x$ and $y$ are chosen at random in the interval $(0, 1)$, what is the probability that the number $\langle x/y \rangle$ is even? Justify your reasoning, and express your answer in the form $r + sc$, where $r$ and $s$ are rational numbers and $c$ is a famous constant.

[Solutions at `http://www.math.uiuc.edu/contests.html`]