2013 UI FRESHMAN MATH CONTEST

Solutions

1. Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and

$$a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1$$

for $n \geq 3$. Find, with proof, a_{2013}.

Solution. Answer: $\boxed{1007^2 (= 1014049)}$.

The answer is not hard to guess by computing the first couple of values of the sequence: $0, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, \ldots$. The odd-indexed terms here are $1, 4, 9, 16, 25, 36, 49, \ldots$, which strongly suggests the formula $(*)$ $a_{2n+1} = (n+1)^2$.

For a formal proof, rewrite the given recurrence as

$$a_n - a_{n-2} = a_{n-1} - a_{n-3} + 1,$$

and iterate to get, for $n \geq 2$,

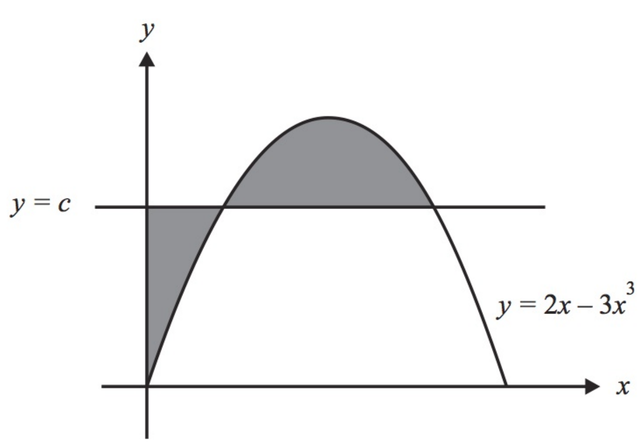
$$a_n - a_{n-2} = a_{n-1} - a_{n-3} + 1 = a_{n-2} - a_{n-4} + 2 = \cdots = a_2 - a_0 + (n-2) = n.$$

Hence

$$\begin{aligned} a_{2n+1} &= a_1 + \sum_{k=1}^{n} (a_{2k+1} - a_{2k-1}) \\ &= 1 + \sum_{k=1}^{n} (2k+1) = 1 + 2\frac{n(n+1)}{2} + n = (n+1)^2. \end{aligned}$$

In particular, $a_{2013} = a_{2 \cdot 1006+1} = 1007^2$.

2. (Problem A1, Putnam 1993) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find, with proof, c so that the areas of the two shaded regions are equal.



Solution. Answer: $\boxed{4/9}$

The difference between the area of the second region and that of the first region is the integral $\int_0^a (2x-3x^3-c)dx$, where a is the second intersection point of the curves $y=c$ and $y=2x-3x^3$. Thus, we need to choose c so that this integral equals zero. Evaluating the integral, we get

$$\int_0^a (2x-3x^3-c)dx = \left[x^2 - \frac{3}{4}x^4 - cx\right]_0^a = a^2 - \frac{3}{4}a^4 - ca,$$

and setting this equal to 0 gives

$$0 = a^2 - \frac{3}{4}a^4 - ca. \tag{1}$$

On the other hand, at $x=a$ the curves $y=c$ and $y=2x-3x^3$ intersect, so we also have

$$c = 2a - 3a^3. \tag{2}$$

Substituting (2) into (1) gives

$$0 = a^2 - \frac{3}{4}a^4 - (2a-3a^3)a = -a^2 + \frac{9}{4}a^4.$$

Thus, $a^2 = (9/4)a^4$, and hence $a = 2/3$ since a must be positive. From (2) we then get $c = 2(2/3) - 3(2/3)^3 = 4/9$.

3. Prove that the product of any 2013 consecutive integers (for example, $1000 \cdot 1001 \cdots 3010 \cdot 3012$) is divisible by 2013! (i.e., by $1 \cdot 2 \cdots 2012 \cdot 2013$).

   Solution. Note that the product of 2013 consecutive integers starting with n equals

$$n(n+1)\dots(n+2012) = \frac{(n+2012)!}{(n-1)!} = 2013!\frac{(n+2012)!}{(n-1)!2013!} = 2013!\binom{n+2012}{2013},$$

   by the definition of binomial coefficients. Since binomial coefficients are integers, this proves that 2013! divides such a product.

4. Let $f(n)$ be the number of n-letter words that can formed with the letters A,B,C,D and such that the letter A occurs an *even* number of times. For example, when $n=1$, there are 3 such words, namely B,C,D, so $f(1)=3$; when $n=2$, there are 10 such words, namely AA,BB,BC,BD,CB,CC,CD,DB,DC,DD, so $f(2)=10$. Find, with proof, a *simple* formula for $f(n)$. (The formula should not involve a summation of more than two terms.)

   Solution. Answer: $\boxed{(1/2)(4^n + 2^n)}$

   The number of such words with exactly k A's is $\binom{n}{k}3^{n-k}$, since there are $\binom{n}{k}$ ways to place the k A's, and 3^{n-k} ways to fill the remaining slots with letters B,C,D. Hence the number of such words with an even number of A's is

$$f(n) = \binom{n}{0}3^n + \binom{n}{2}3^{n-2} + \binom{n}{4}3^{n-4} + \cdots, \tag{1}$$

   where the sum is over all terms $\binom{n}{k}3^{n-k}$ with $0 \le k \le n$ and k even. To get the required "simple" formula, we need to evaluate the sum in (1). To do this, consider the identities

$$(1+3)^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k}, \tag{2}$$

$$(-1+3)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k} \tag{3}$$

which follow from the binomial theorem. Adding (2) and (3) gives twice the sum on the right of (1). Hence

$$f(n) = \frac{1}{2}\left((1+3)^n + (-1+3)^n\right) = \frac{1}{2}\left(4^n + 2^n\right).$$

Alternative solution (Hao Gao): Use recurrences and induction. An $(n+1)$-letter word with an *even* number of A's must be either of the form (i) WX, where X is one of the letters B,C,D and W is a word with n letters and an *even* number of A's, or of the form (ii) WA, where W is a word with n letters and an *odd* number of A's. There are $3f(n)$ words of type (i), and $4^n - f(n)$ words of type (ii), so we get the recurrence

$$f(n+1) = 3f(n) + (4^n - f(n)) = 2f(n) + 4^n. \tag{4}$$

Now use induction and (4) to prove the desired formula, i.e., $(*)$ $f(n) = 2^{2n-1} + 2^{n-1}$. The base case $n = 1$ is trivial: We have $f(1) = 3$ and $2^{2\cdot 1-1} + 2^{1-1} = 3$, so $(*)$ holds. Now let n be given and assume $(*)$ holds for this n-value. Then, by (4), $f(n+1) = 2\cdot(2^{2n-1} + 2^{n-1}) + 4^n = 2^{2n+1} + 2^n$, which proves $(*)$ for the next value, $n+1$, and completes the induction.

5. Show that the number

$$N = 1^{2013} + 2^{2013} + \cdots + 2013^{2013}$$

is divisible by 2013^2.

Solution. We show more generally that, for any *odd* positive integer n, the sum $1^n + 2^n + \cdots + n^n$ is divisible by n^2.

Since n^n is divisible by n^2, it suffices to consider the sum $1^n + 2^n + \cdots + (n-2)^n + (n-1)^n$. Since n is odd, we can match up k^n with $(n-k)^n$, for $k = 1, 2, \ldots (n-1)/2$, and it suffices to show that $k^n + (n-k)^n$ is divisible by n^2 for each k. Expanding $(n-k)^n$ and reducing modulo n^2, we see that

$$\begin{aligned}(n-k)^n &= (-k)^n + \binom{n}{k} n^1(-k)^{n-1} + \sum_{i=2}^{n} \binom{n}{k} n^i(-k)^{n-i} \\ &\equiv (-1)^n k^n = -k^n \bmod n^2,\end{aligned}$$

since $n$is odd. It follows that $k^n + (n-k)^n \equiv 0 \bmod n^2$, so $k^n + (n-k)^n$ is divisible by n^2, as claimed.

6. Show that the polynomial

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}$$

(where n is a natural number) has no real roots.

Solution. Since $P_n(x)$ has even degree with positive leading coefficient, we have $P_n(x) \to \infty$ as $x \to \pm\infty$. Thus, $P_n(x)$ attains a minimum value at some x_0, and it suffices to show that $P_n(x_0) > 0$. Since x_0 is a global and local minimum for P_n, we have $P_n'(x_0) = 0$. On the other hand, computing $P_n'(x)$ directly gives

$$\begin{aligned}P_n'(x) &= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{(2n)x^{2n-1}}{(2n)!} \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} = P_n(x) - \frac{x^{2n}}{(2n)!}.\end{aligned}$$

Hence

$$P_n(x_0) = P_n'(x_0) + \frac{x_0^{2n}}{(2n)!}, = \frac{x_0^{2n}}{(2n)!} > 0$$

as claimed.

Alternative solution (Xinlun Li): Use Taylor's Theorem.

First note that, if $x \geq 0$, then $P_n(x) \geq 1$, so $P_n(x)$ has no zeros with $x \geq 0$. Thus, it remains to consider the case when $x < 0$. In this case, we apply Taylor's formula of order $2n$ to the function e^x, to get

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} + \frac{y^{2n+1}}{(2n+1)!} = P_n(x) + \frac{y^{2n+1}}{(2n+1)!} \tag{1}$$

for some y between 0 and x. Since $x < 0$, we must have $y < 0$ as well. Hence the last term in (1), i.e., $y^{2n+1}/(2n+1)!$, is negative, so we get $e^x < P_n(x)$ for $x < 0$. Since $e^x > 0$, this implies $P_n(x) > 0$ for $x < 0$, so $P_n(x)$ cannot have a negative zero either.