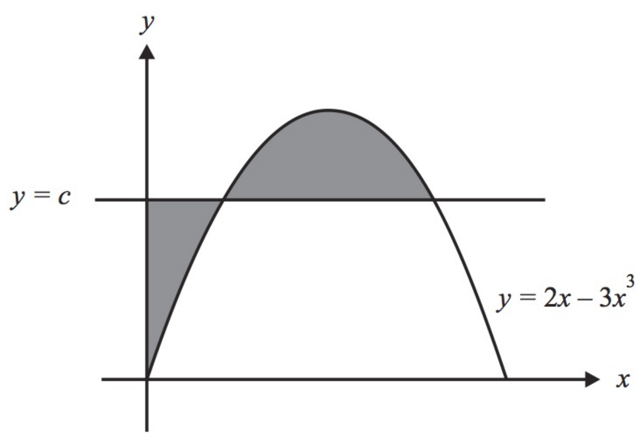
2013 UI FRESHMAN MATH CONTEST
September 23, 2013, 5 pm - 7 pm

1. Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and

$$a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1$$

   for $n \geq 3$. Find, with proof, a_{2013}.

2. (Problem A1, Putnam 1993) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find, with proof, c so that the areas of the two shaded regions are equal.



3. Prove that the product of any 2013 consecutive integers (for example, $1000 \cdot 1001 \cdots 3010 \cdot 3012$) is divisible by 2013! (i.e., by $1 \cdot 2 \cdots 2012 \cdot 2013$).

4. Let $f(n)$ be the number of n-letter words that can formed with the letters A,B,C,D and such that the letter A occurs an *even* number of times. For example, when $n = 1$, there are 3 such words, namely B,C,D, so $f(1) = 3$; when $n = 2$, there are 10 such words, namely AA,BB,BC,BD,CB,CC,CD,DB,DC,DD, so $f(2) = 10$. Find, with proof, a *simple* formula for $f(n)$. (The formula should not involve a summation of more than two terms.)

5. Show that the number

$$N = 1^{2013} + 2^{2013} + \cdots + 2013^{2013}$$

   is divisible by 2013^2.

6. Show that the polynomial

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}$$

   (where n is a natural number) has no real roots.

[Solutions at `http://www.math.uiuc.edu/contests.html`]