2012 UI FRESHMAN MATH CONTEST

Solutions

1. Determine, with proof, whether there exists a power of 2 whose decimal representation ends in the digits 2012.

   Solution. The answer is no. To see this, note that a number ending in 2012 must be of the form $10000k + 2012 = 4(2500k + 503)$ for some nonnegative integer k. Since $2500k + 503$ is odd, such a number must have an odd prime factor and therefore cannot be equal to a power of 2.

2. [A1, Putnam 1985] Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of subsets of $\{1, 2, \dots, 2012\}$ with the following properties:

   (i) Each of the integers $1, 2, \dots, 2012$ belongs to at least one of the sets A_1, A_2, A_3.

   (ii) None of the integers $1, 2, \dots, 2012$ belongs to all three of the sets A_1, A_2, A_3.

   Solution. Encode the memberships of an element in $\{1, 2, \dots, 2012\}$ as a triple (a_1, a_2, a_3) with $a_i = 1$ if the integer belongs to A_i, and 0 otherwise. For each such element there are 6 such encodings that satisfy the constraints (i) and (ii): $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 0, 1)$. Since there are 2012 elements, the total number of possible encodings, and hence the total number of set triples (A_1, A_2, A_3) satisfying (i) and (ii), is $\boxed{6^{2012}}$.

3. Prove that 2012 can be represented in the form

   $$2012 = \pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm m^2$$

   for some positive integer m and a suitable choice of the $\pm$ signs. (For example, $4 = -1^2 - 2^2 + 3^2$ is a representations of the required form for the number 4 with $m = 3$ and $-$, $-$, $+$ as the sequence of $\pm$ signs. Note that each of the squares $1^2, 2^2, 3^2, \dots, m^2$ must be used in this representation.)

   Solution. Note that, for any positive integer k, $k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2 = 4$. Applying this with $k = 1 + 4h$, where $h = 0, 1, 2, \dots, 502$, we get

   $$\sum_{h=0}^{502} \left((4h+1)^2 - (4h+2)^2 - (4h+3)^2 + (4h+4)^2\right) = \sum_{h=0}^{502} 4 = 503 \cdot 4 = 2012.$$

   This gives a representation of the desired form for 2012.

4. Evaluate the integral

   $$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx$$

   (Hint: No special integration techniques needed!)

   Solution. We exploit the symmetry of the integrand about $x = 3$. Let $f(x)$ denote the integrand. Then it is easily checked that $f(3 - y) = 1 - f(3 + y)$. Hence, making the change of variables $x = 3 + y$, the integral becomes $\int_{-1}^{1} (f(3+y) dy = \int_0^1 f(3-y) + f(3+y)) dy = \int_0^1 1 dy = 1$.

5. Suppose each point in the plane is colored either orange or blue. Define D_O, the set of "orange distances", as the set of positive real numbers d for which there exist two orange-colored points whose distance is exactly d, and let D_B, the set of "blue distances", be defined analogously with respect to blue-colored points. Prove that at least one of the two sets D_O and D_B contains all positive real numbers.

   Solution. Suppose, to the contrary, that there exist positive distances d_O and d_B such that no two orange points have distance d_O and no two blue points have distance d_B. Without loss of generality, we may assume $d_O \leq d_B$. Our assumption implies, in particular, that there exists at least one blue point, say B. Now draw a circle of radius d_B around B. Since no two blue points have distance d_B, every point on the circle must be colored orange. Since $d_O \leq d_B$, there exist two points on the circle that are distance d_O apart. This contradicts the assumption that no two orange points have distance d_O. Hence, at least one of the two colors contains pairs of points at every distance.

6. [A2, Putnam 1986] Determine, with proof, the rightmost digit (in decimal) of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$ (where $\lfloor x \rfloor$ denotes the largest integer $\leq x$).

Solution. Let x denote the number in brackets. Writing x as $10^{19,900}(1+r)^{-1}$, with $r = 3 \cdot 10^{-100}$ and expanding $(1+r)^{-1}$ into a geometric series we get

$$x = 10^{19,900} \sum_{n=0}^{\infty} (-r)^n = \sum_{n=0}^{\infty} (-1)^n 3^n 10^{19,900-100n}.$$

In the last sum, all terms with $n < 199$ are all divisible by 10 and the term $n = 199$ equals $(-3)^{199}$. Also, since the series is alternating with decreasing terms, the sum over the remaining terms (i.e., those with $n \geq 200$) is positive and bounded from above by the first of these terms, i.e., $3^{200}10^{-100} = (9/10)^{100}$, which is less than 1. Thus, $\lfloor x \rfloor = 10k + N$, where k is an integer and $N = (-3)^{199}$. Hence the rightmost digit of $\lfloor x \rfloor$ is equal to the rightmost digit of $10k + N$, which in turn is equal to the remainder of N modulo 10. Since $(-3)^4 = 81 \equiv 1$ modulo 10, we have $N = (-3)^{199} = -27 \cdot 81^{49} \equiv -27 \equiv 3$ modulo 10. Therefore the rightmost digit of $[x]$ is $\boxed{3}$.