1. Determine, with proof, whether there exists a power of 2 whose decimal representation ends in the digits 2012.

Solution. The answer is no. To see this, note that a number ending in 2012 must be of the form \(10000k + 2012 = 4(2500k + 503)\) for some nonnegative integer \(k\). Since \(2500k + 503\) is odd, such a number must have an odd prime factor and therefore cannot be equal to a power of 2.

2. [A1, Putnam 1985] Determine, with proof, the number of ordered triples \((A_1, A_2, A_3)\) of subsets of \(\{1, 2, \ldots, 2012\}\) with the following properties:

(i) Each of the integers \(1, 2, \ldots, 2012\) belongs to at least one of the sets \(A_1, A_2, A_3\).

Solution. Encode the memberships of an element in \(\{1, 2, \ldots, 2012\}\) as a triple \((a_1, a_2, a_3)\) with \(a_i = 1\) if the integer belongs to \(A_i\), and 0 otherwise. For each such element there are 6 such encodings that satisfy the constraints (i) and (ii): \((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 0, 1)\). Since there are 2012 elements, the total number of possible encodings, and hence the total number of set triples \((A_1, A_2, A_3)\) satisfying (i) and (ii), is \(6^{2012}\).

3. Prove that 2012 can be represented in the form

\[2012 = \pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm m^2\]

for some positive integer \(m\) and a suitable choice of the \(\pm\) signs. (For example, \(4 = -1^2 - 2^2 + 3^2\) is a representations of the required form for the number 4 with \(m = 3\) and \(-, -, +\) as the sequence of \(\pm\) signs. Note that each of the squares \(1^2, 2^2, 3^2, \ldots, m^2\) must be used in this representation.)

Solution. Note that, for any positive integer \(k\), \(k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2 = 4\). Applying this with \(k = 1 + 4h\), where \(h = 0, 1, 2, \ldots, 502\), we get

\[
\sum_{h=0}^{502} \left( (4h + 1)^2 - (4h + 2)^2 - (4h + 3)^2 + (4h + 4)^2 \right) = \sum_{h=0}^{502} 4 = 503 \cdot 4 = 2012.
\]

This gives a representation of the desired form for 2012.

4. Evaluate the integral

\[
\int_{\frac{2}{7}}^{1} \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x) + \sqrt{\ln(x + 3)}}} \, dx
\]

(Hint: No special integration techniques needed!)

Solution. We exploit the symmetry of the integrand about \(x = 3\). Let \(f(x)\) denote the integrand. Then it is easily checked that \(f(3 - y) = 1 - f(3 + y)\). Hence, making the change of variables \(x = 3 + y\), the integral becomes

\[
\int_{-1}^{0} (f(3 + y)dy = \int_{0}^{1} f(3 - y) + f(3 + y))dy = \int_{0}^{1} 1dy = 1.
\]

5. Suppose each point in the plane is colored either orange or blue. Define \(D_O\), the set of “orange distances”, as the set of positive real numbers \(d\) for which there exist two orange-colored points whose distance is exactly \(d\), and let \(D_B\), the set of “blue distances”, be defined analogously with respect to blue-colored points. Prove that at least one of the two sets \(D_O\) and \(D_B\) contains all positive real numbers.

Solution. Suppose, to the contrary, that there exist positive distances \(d_O\) and \(d_B\) such that no two orange points have distance \(d_O\) and no two blue points have distance \(d_B\). Without loss of generality, we may assume \(d_O \leq d_B\). Our assumption implies, in particular, that there exists at least one blue point, say \(B\). Now draw a circle of radius \(d_B\) around \(B\). Since no two blue points have distance \(d_B\), every point on the circle must be colored orange. Since \(d_O \leq d_B\), there exist two points on the circle that are distance \(d_O\) apart. This contradicts the assumption that no two orange points have distance \(d_O\). Hence, at least one of the two colors contains pairs of points at every distance.
6. [A2, Putnam 1986] Determine, with proof, the rightmost digit (in decimal) of \( \left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor \) (where \([x]\) denotes the largest integer \( \leq x\)).

**Solution.** Let \( x \) denote the number in brackets. Writing \( x \) as \( 10^{19.900}(1 + r)^{-1} \), with \( r = 3 \cdot 10^{-100} \) and expanding \((1 + r)^{-1}\) into a geometric series we get

\[
x = 10^{19.900} \sum_{n=0}^{\infty} (-r)^n = \sum_{n=0}^{\infty} (-1)^n 3^n 10^{19.900 - 100n}.
\]

In the last sum, all terms with \( n < 199 \) are all divisible by 10 and the term \( n = 199 \) equals \((-3)^{199}\). Also, since the series is alternating with decreasing terms, the sum over the remaining terms (i.e., those with \( n \geq 200 \)) is positive and bounded from above by the first of these terms, i.e., \( 3^{200} 10^{-100} = (9/10)^{100} \), which is less than 1. Thus, \([x] = 10k + N\), where \( k \) is an integer and \( N = (-3)^{199} \). Hence the rightmost digit of \([x]\) is equal to the rightmost digit of \(10k + N\), which in turn is equal to the remainder of \(N\) modulo 10. Since \((-3)^4 = 81 \equiv 1 \) modulo 10, we have \( N = (-3)^{199} = -27 \cdot 81^{49} \equiv -27 \equiv 3 \) modulo 10. Therefore the rightmost digit of \([x]\) is \(3\).