2012 UI FRESHMAN MATH CONTEST

1. Determine, with proof, whether there exists a power of 2 whose decimal representation ends in the digits 2012.

2. [A1, Putnam 1985] Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of subsets of $\{1, 2, \ldots, 2012\}$ with the following properties:

   (i) Each of the integers $1, 2, \ldots, 2012$ belongs to at least one of the sets A_1, A_2, A_3.

   (ii) None of the integers $1, 2, \ldots, 2012$ belongs to all three of the sets A_1, A_2, A_3.

3. Prove that 2012 can be represented in the form

$$2012 = \pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm m^2$$

   for some positive integer m and a suitable choice of the $\pm$ signs. (For example, $4 = -1^2 - 2^2 + 3^2$ is a representations of the required form for the number 4 with $m = 3$ and $-, -, +$ as the sequence of $\pm$ signs. Note that each of the squares $1^2, 2^2, 3^2, \ldots, m^2$ must be used in this representation.)

4. Evaluate the integral

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx$$

   (Hint: No special integration techniques needed!)

5. Suppose each point in the plane is colored either orange or blue. Define D_O, the set of "orange distances", as the set of positive real numbers d for which there exist two orange-colored points whose distance is exactly d, and let D_B, the set of "blue distances", be defined analogously with respect to blue-colored points. Prove that at least one of the two sets D_O and D_B contains all positive real numbers.

6. [A2, Putnam 1986] Determine, with proof, the rightmost digit (in decimal) of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$ (where $\lfloor x \rfloor$ denotes the largest integer $\leq x$).

[Solutions at `http://www.math.uiuc.edu/contests.html`]