2011 U OF I FRESHMAN MATH CONTEST

Solutions

1. Let $x = 0.12345678910111213\ldots$ be the number whose decimal expansion consists of the sequence of natural numbers written next to each other.

   (a) Determine the 2011th digit after the decimal point of x.

   (b) Prove that x is irrational.

   Solution. (a) The 2011th digit is $\boxed{7}$. To show this, we keep track of the positions occupied by the integers $1, 2, 3, \ldots$ that form the sequence of digits. The 9 single digit integers $1, 2, \ldots, 9$ occupy positions $1, 2, \ldots, 9$ in the sequence; the 90 two digit integers $10, 11, \ldots, 99$, occupy positions $9 + 1 = 10, \ldots, 9 + 2 \cdot 90 = 189$; the 900 three digit integers $100, 101, \ldots, 999$ occupy positions $189 + 1, \ldots, 189 + 3 \cdot 900 = 2889$. The latter range contains 2011, the position we are interested in; more precisely, since $189 + 3 \cdot 607 = 2010$, the integer $607 + 99 = 706$ occupies positions $2008, 2009, 2010$, and 707 occupies positions $2011, 2012, 2013$. Hence the digit in position 2011 is 7, as claimed..

   (b) We argue by contradiction. Suppose x is rational. Then its decimal expansion is ultimately periodic. Let p denote the period of this expansion. Now consider a block $B = 0 \ldots 0$ consisting of p consecutive digits 0. Since any integer of the form 10^k with $k \geq p$ contains p consecutive 0's, this block must occur infinitely often in the decimal expansion of x. By our assumption that this expansion is ultimately periodic with period p, this implies that B must be the repeating period block, which means that the sequence consists of all 0's from some point onwards. But this clearly contradicts the construction of the sequence.

2. Find, with proof, a simple formula for the sum

$$S_n = \sum_{k=1}^{n} \frac{k}{(k+1)!}.$$

   Solution. The desired formula is $(*)$ $\boxed{S_n = 1 - 1/(n+1)!}$.

   We prove $(*)$ by induction: In the base case $n = 1$ we have $S_1 = 1/(1+1)! = 1 - 1/2!$, so the formula holds in this case. For the induction step, assuming the validity of the formula for S_n. Then

$$S_{n+1} = S_n + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!},$$

   which is the desired formula for S_{n+1}, completing the induction.

3. There are 92 airports in Illinois. Suppose that from each of these airports a plane takes off and flies to the nearest neighboring airport. Assuming the mutual distances between the airports are all distinct prove that there is no airport at which more than five planes land.

   Solution. We argue by contradiction. Suppose that there is an airport, say A, at which 6 (or more) planes land. Then for two of the six originating airports, say B and C, the angle formed by the routes from these airports to A is $\leq 360/6 = 60$ degrees. Thus, in the triangle ABC, the angle at A is ≤ 60 degrees. But then one of the other two angles in this triangle must be ≥ 60 degrees. Without loss of generality, assume that the angle at B is ≥ 60 degrees. By the sine law, it follows that the side opposing B is greater or equal to the side opposing A, i.e., $|AC| \geq |BC|$. Since the mutual distances between the airports were assumed to be distinct, we must have strict inequality, i.e., $|AC| > |BC|$. But then B is closer to C than A, contradicting our assumption that A is the nearest airport to C.

4. Find, with proof, a simple formula for the sum

$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-k}.$$

   Solution. Let $S(n)$ denote the given sum. We will show by induction that $(*)$ $\boxed{S(n) = 2^n}$.

For $n=1$, we have $S(1)=\binom{1}{0}+\binom{2}{1}2^{-1}=2$, so (∗) holds in this case. Now let $n\ge 1$ and suppose (∗) holds for this n. Then, using the recurrence formula for binomial coefficients, we get

$$\begin{aligned} S(n+1) &= \sum_{k=0}^{n+1}\binom{n+1+k}{k}2^{-k} \\ &= \sum_{k=0}^{n+1}\binom{n+k}{k}2^{-k}+\sum_{k=1}^{n+1}\binom{n+k}{k-1}2^{-k} \\ &= \sum_{k=0}^{n}\binom{n+k}{k}2^{-k}+\binom{2n+1}{n+1}2^{-n-1}+\sum_{h=0}^{n}\binom{n+1+h}{h}2^{-h-1} \\ &= S(n)+\binom{2n+1}{n+1}2^{-n-1}+\frac{1}{2}\left(S(n+1)-\binom{2n+2}{n+1}2^{-n-1}\right) \\ &= S(n)+\frac{1}{2}S(n+1)+\left(\binom{2n+1}{n+1}-\frac{1}{2}\binom{2n+2}{n+1}\right)2^{-n-1}. \end{aligned}$$

Since

$$\binom{2n+2}{n+1}=\binom{2n+1}{n+1}+\binom{2n+1}{n}=2\binom{2n+1}{n+1},$$

the last term is zero, so we have $S(n+1)=S(n)+(1/2)S(n+1)$, and hence $S(n+1)=2S(n)=2\cdot 2^n=2^{n+1}$. This proves the formula for $n+1$ and completes the induction.

5. Given positive integers r and s, let $f(r,s)$ denote the number of 4-tuples of positive integers (a,b,c,d) such that the least common multiple of any **three** of these integers is equal to 3^r7^s (i.e., such that $[a,b,c]=[a,b,d]=[a,c,d]=[b,c,d]=3^r7^s$, where $[\dots]$ denotes the least common multiple). Find, with proof, a simple formula for $f(r,s)$.

Solution. The desired formula is $\boxed{f(r,s)=(1+4r+6r^2)(1+4s+6s^2)}$. To prove this, note first that the four integers can have only the primes 3 and 7 in their prime factorization and thus can be written as $3^{r_i}7^{s_i}$, $i=1,2,3,4$, with nonnegative integers r_i and s_i. The condition that any three of these numbers have least common multiple 3^r7^s is then equivalent to the condition that (∗) *the largest value among* r_1,r_2,r_3,r_4 *is equal to* r *and is attained by at least two of the* r_i*'s*, along with an analogous condition on the s_i's. The desired count $f(r,s)$ is then given by $f(r,s)=g(r)g(s)$, where $g(r)$ is the number of tuples (r_1,r_2,r_3,r_4) satisfying (∗).

To obtain a formula for $g(r)$, note that any tuple (r_1,r_2,r_3,r_4) satisfying (∗) must fall into exactly one of the following cases: (i) all r_i are equal to r; (ii) three of the numbers r_i are equal to r and the other number is $<r$; and (iii) two of the numbers r_i are equal to r and the other two numbers are $<r$.

The number of tuples falling into these cases are 1 for case (i); $\binom{4}{1}r$ for case (ii); and $\binom{4}{2}r^2$ for case (iii). Thus, the total number of ways to pick a tuple (r_1,r_2,r_3,r_4) satisfying the above condition is $g(r)=1+\binom{4}{1}r+\binom{4}{2}r^2=1+4r+6r^2$. Hence $f(r,s)=g(r)g(s)=(1+4r+6r^2)(1+4s+6s^2)$, as claimed.

6. Find, with proof, the sum of the infinite series

$$\frac{4}{4^2-1}+\frac{4^2}{4^4-1}+\frac{4^4}{4^8-1}+\frac{4^8}{4^{16}-1}+\cdots$$

Solution. Let

$$f(x)=\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}}.$$

The given series is $f(1/4)$. Writing each term in this series as a product $x^{2^n}(1-x^{2^{n+1}})^{-1}$ and expanding the second factor into a geometric series, we get

$$f(x)=\sum_{n=0}^{\infty}x^{2^n}\sum_{k=0}^{\infty}x^{2^{n+1}k}=\sum_{n,k=0}^{\infty}x^{2^n+2^{n+1}k}=\sum_{n,k=0}^{\infty}x^{2^n(1+2k)}.$$

In the last series, the exponents $2^n(1+2k)$ are positive integers, and each positive integer occurs exactly once as such an exponent. Hence the last series is equal to $\sum_{m=1}^{\infty}x^m=x/(1-x)$, and so $f(x)=x/(1-x)$. The value of the given sum is therefore $f(1/4)=(1/4)/(1-1/4)=\boxed{1/3}$.