1999 UIUC Undergraduate Math Contest

Solutions

Problem 1.

Let a_n denote the integer closest to $\sqrt{n}$. (For example, $a_1 = a_2 = 1$ and $a_3 = a_4 = 2$ since $\sqrt{1} = 1$, $\sqrt{2} = 1.41\ldots$, $\sqrt{3} = 1.73\ldots$, and $\sqrt{4} = 2$.) Evaluate the sum

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{1980}}.$$

Solution. For any positive integer k, a_n is equal to k, if and only if $\sqrt{n}$ lies between $k - 1/2$ and $k + 1/2$, i.e., if and and only if n lies between $k^2 - k + 1/4$ and $k^2 + k + 1/4$. Since there are exactly $2k$ integer values in this range, and since $1980 = 44 \cdot 45 = 44^2 + 44$, it follows that $S = \sum_{k=1}^{44} (1/k) \cdot 2k = 88$.

Problem 2.

Let ABC be a triangle, and let BD and CE denote the angle-bisectors at B and C. Show that if BD and CE have the same length, then the triangle is isosceles (that is, the sides AB and AC have the same length).

Solution. Let $a = BC$, $b = AC$, $c = AB$ denote the three sides of the triangle, β and γ the angles of the triangle at B and C, and $d = BD = CE$ the (common) length of the angle-bisectors at these points. The area $\mathbf{A}$ of the triangle ABC is, on the one hand, $\mathbf{A} = (1/2)ac\sin\beta$. On the other hand, splitting ABC into the triangles BCD and BDA, which have areas $(1/2)ad\sin(\beta/2)$ and $(1/2)dc\sin(\beta/2)$, respectively, we obtain $\mathbf{A} = (1/2)d(a + c)\sin(\beta/2)$. Setting the two expressions for A equal and using the double angle formula $\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$, it follows that (1) $2\cos(\beta/2) = d(1/a + 1/c)$. Similarly, interchanging the roles of B and C, we obtain (2) $2\cos(\gamma/2) = d(1/a + 1/b)$. If we now assume that b and c are not equal, say (without loss of generality) $b < c$, then $\beta < \gamma$ and so $\cos(\beta/2) > \cos(\gamma/2)$. However, by (1) and (2) this would imply $1/c > 1/b$, contradicting the assumption $b < c$. Hence b and c must be equal as claimed.

Problem 3.

Let a sequence $\{x_n\}$ be given by $x_1 = 1$ and $x_{n+1} = x_n^2 + x_n$ for $n = 1, 2, 3, \ldots$. Let $y_n = 1/(1 + x_n)$ and let $S_n = \sum_{k=1}^{n} y_k$ and $P_n = \prod_{k=1}^{n} y_k$ denote, respectively, the sum and the product of the first n terms of the sequence $\{y_k\}$. Evaluate $P_n + S_n$ for $n = 1, 2, 3, \ldots$.

Solution. From the given recurrence we obtain $x_{n+1} = x_n/y_n$, so that $y_n = x_n/x_{n+1}$ for all n. Hence $P_n = \prod_{k=1}^{n}(x_k/x_{k+1}) = x_1/x_{n+1} = 1/x_{n+1}$ for all n. Moreover, from the identity

$$y_n = \frac{1}{1 + x_n} = \frac{1}{x_n} - \frac{1}{(1 + x_n)x_n} = \frac{1}{x_n} - \frac{1}{x_{n+1}},$$

we see that $S_n = \sum_{k=1}^n (1/x_k - 1/x_{k+1}) = 1/x_1 - 1/x_{n+1} = 1 - 1/x_{n+1}$. Hence $P_n + S_n = 1$ for all n.

Problem 4.

Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{\,x_n}$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Solution. Since $x_1 = \sqrt{2} < 2$ and if $x_n < 2$ then $x_{n+1} = \sqrt{2}^{\,x_n} < \sqrt{2}^{\,2} = 2$, it follows by induction that (1) $x_n < 2$ for all n. Thus, the sequence $\{x_n\}$ is bounded from above. Next let $f(x) = \sqrt{2}^{\,x} - x$. Then $f'(x) = \sqrt{2}^x \log\sqrt{2} - 1 < 2\log\sqrt{2} - 1 < 0$ for $x < 2$, so $f(x)$ is decreasing for $x < 2$, and since $f(2) = 0$, this implies $f(x) > 0$, or equivalently $\sqrt{2}^x > x$, for $x < 2$. In view of (1), it follows that $x_{n+1} = \sqrt{2}^{\,x_n} > x_n$ for all n. Hence the sequence $\{x_n\}$ is monotone increasing and bounded from above and therefore must be convergent. Let L denote the limit of this sequence. By (1) we have (2) $L = \lim_{n\to\infty} x_n \leq 2$, and letting $n \to \infty$ on both sides of the recurrence $x_{n+1} = \sqrt{2}^{\,x_n}$, we obtain $L = \sqrt{2}^L$ or (3) $f(L) = 0$. Since $f(2) = 0$, $L = 2$ is a solution to (3). Moreover, $L = 2$ is the only solution satisfying (2), since $f(x)$ is decreasing for $x < 2$. Hence the limit of the sequence $\{x_n\}$ is 2.

Problem 5.

Prove that the series

$$\frac{1}{1} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \dots$$

converges and evaluate its sum.

Solution. Let S_n denote the sum of the first n terms of this sequence. Then

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{[n/3]} \frac{3}{3k} = \sum_{k=[n/3]+1}^{n} \frac{1}{k}.$$

Let T_n denote the latter sum. Comparing this sum with an integral we see that

$$\log 3 - \log(1 + \frac{3}{n}) = \int_{n/3+1}^{n} \frac{1}{x} dx \leq T_n \leq \int_{n/3-1}^{n} \frac{1}{x} dx = \log 3 - \log(1 - \frac{3}{n})$$

Since $\log(1 \pm 3/n) \to 0$ as $n \to \infty$, we conclude that T_n, and therefore S_n, converges, and has limit $\log 3$. Hence the given infinite series converges with sum $\log 3$.

Problem 6.

Given positive integers n and m with $n \geq 2m$, let $f(n, m)$ be the number of binary sequences of length n (i.e., strings $a_1a_2\dots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. Find a simple formula for $f(n, m)$.

Solution. Every sequence of the required form can be written as $B_1C_101B_2C_201\ldots01B_{m+1}C_{m+1}$, where each B_i is a block of 1's and each C_i a block of 0's, with empty blocks being allowed, and the sum of the lengths of the blocks B_i and C_i is $n-2m$. Moreover, the sequence is uniquely determined by the $(2m+2)$-tuple (1) $(b_1, c_1, b_2, c_2, \ldots, b_{m+1}, c_{m+1})$ where b_i and c_i denote the number of elements in the blocks B_i and C_i, respectively. Conversely, any tuple of the form (1) with nonnegative integers b_i and c_i satisfying $\sum_{i=1}^{m+1}(b_i+c_i)=(n-2m)$ determines a sequence of the required type. Hence the number of such sequences is equal to the number of ways one can write $2n-m$ as a sum of $2m+2$) nonnegative integers, with order taken into account. The latter problem is equivalent to counting the number of ways of choosing $2n-m$ donuts from $2m+2$ varieties, a well-known combinatorial problem whose answer is given by the binomial coefficient $\binom{a}{b}$ with $a=(n-2m)+(2m+2)-1=n+1$ and $b=(2m+2)-1=2m+1$. Hence $f(n,m)=\binom{n+1}{2m+1}$.