1999 UIUC Undergraduate Math Contest

Problem 1.

Let a_n denote the integer closest to $\sqrt{n}$. (For example, $a_1 = a_2 = 1$ and $a_3 = a_4 = 2$ since $\sqrt{1} = 1$, $\sqrt{2} = 1.41\ldots$, $\sqrt{3} = 1.73\ldots$, and $\sqrt{4} = 2$.) Evaluate the sum

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{1980}}.$$

Problem 2.

Let ABC be a triangle, and let BD and CE denote the angle-bisectors at B and C. Show that if BD and CE have the same length, then the triangle is isosceles (that is, the sides AB and AC have the same length).

Problem 3.

Let a sequence $\{x_n\}$ be given by $x_1 = 1$ and $x_{n+1} = x_n^2 + x_n$ for $n = 1, 2, 3, \ldots$. Let $y_n = 1/(1 + x_n)$ and let $S_n = \sum_{k=1}^n y_k$ and $P_n = \prod_{k=1}^n y_k$ denote, respectively, the sum and the product of the first n terms of the sequence $\{y_k\}$. Evaluate $P_n + S_n$ for $n = 1, 2, 3, \ldots$.

Problem 4.

Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{\,x_n}$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Problem 5.

Prove that the series

$$\frac{1}{1} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \ldots$$

converges and evaluate its sum.

Problem 6.

Given positive integers n and m with $n \geq 2m$, let $f(n, m)$ be the number of binary sequences of length n (i.e., strings $a_1 a_2 \ldots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. Find a simple formula for $f(n, m)$.