1999 UIUC Undergraduate Math Contest

Problem 1.
Let $a_n$ denote the integer closest to $\sqrt{n}$. (For example, $a_1 = a_2 = 1$ and $a_3 = a_4 = 2$ since $\sqrt{1} = 1$, $\sqrt{2} = 1.41\ldots$, $\sqrt{3} = 1.73\ldots$, and $\sqrt{4} = 2$.) Evaluate the sum

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{1980}}.$$ 

Problem 2.
Let $ABC$ be a triangle, and let $BD$ and $CE$ denote the angle-bisectors at $B$ and $C$. Show that if $BD$ and $CE$ have the same length, then the triangle is isosceles (that is, the sides $AB$ and $AC$ have the same length).

Problem 3.
Let a sequence $\{x_n\}$ be given by $x_1 = 1$ and $x_{n+1} = x^2_n + x_n$ for $n = 1, 2, 3,\ldots$. Let $y_n = 1/(1 + x_n)$ and let $S_n = \sum_{k=1}^n y_k$ and $P_n = \prod_{k=1}^n y_k$ denote, respectively, the sum and the product of the first $n$ terms of the sequence $\{y_k\}$. Evaluate $P_n + S_n$ for $n = 1, 2, 3,\ldots$.

Problem 4.
Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2} x^n$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Problem 5.
Prove that the series

$$\frac{1}{1} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \cdots$$

converges and evaluate its sum.

Problem 6.
Given positive integers $n$ and $m$ with $n \geq 2m$, let $f(n, m)$ be the number of binary sequences of length $n$ (i.e., strings $a_1a_2\ldots a_n$ with each $a_i$ either 0 or 1) that contain the block 01 exactly $m$ times. Find a simple formula for $f(n, m)$. 