1998 UIUC Undergraduate Math Contest

Solutions

Problem 1.

A sequence $a_0, a_1, a_2 \dots$ of real numbers is defined recursively by

$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad (n = 0, 1, 2, \dots).$$

Find a general formula for a_n.

Solution.

Set $b_n = 1/a_n$. The given recurrence then takes the form

$$b_0 = 1, \quad b_{n+1} = b_n + n \quad (n = 0, 1, 2, \dots).$$

Iterating this recurrence we obtain, for $n = 1, 2, \dots,$

$$b_n = b_{n-1} + (n-1) = \dots = b_0 + \sum_{k=0}^{n-1} k = 1 + \frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n + 1.$$

Hence

$$a_n = \frac{1}{b_n} = \left(\frac{1}{2}n^2 - \frac{1}{2}n + 1\right)^{-1} \quad (n = 1, 2, \dots).$$

Problem 2.

Evaluate $\sum_{k=1}^{n} k2^{k-1}$ for $n = 1, 2, \dots$

Solution.

Set $f(x) = \sum_{k=0}^{n} x^k$. Then $f'(x) = \sum_{k=1}^{n} kx^{k-1}$, so the given sum is equal to $f'(2)$. On the other hand, summing the geometric series in the definition of $f(x)$ gives $f(x) = (x^{n+1} - 1)/(x - 1)$, and differentiating this function we obtain

$$f'(x) = \frac{(x-1)(n+1)x^n - (x^{n+1} - 1)}{(x-1)^2}.$$

Hence

$$\sum_{k=1}^{n} k2^{k-1} = f'(2) = (n+1)2^n - 2^{n+1} + 1 = (n-1)2^n + 1.$$

Problem 3.

Given a nonempty finite set A of real numbers, let $m(A)$ denote the maximal element of A. For $n = 1, 2, \dots$, let $f(n)$ be the sum of $m(A)$, where A runs over the $2^n - 1$ non-empty subsets of the set $\{1, 2, \dots, n\}$. Give an explicit formula for $f(n)$.

Solution.

We have $f(n) = \sum_{k=1}^{n} kN(k)$, where $N(k)$ denotes the number of non-empty subsets of $\{1, 2, \ldots, n\}$ whose maximal element is k. Clearly, $N(1) = 1$, since the set $\{1\}$ is the only set of positive integers with maximal element 1. For $k \geq 2$, the sets counted by $N(k)$ are exactly those of the form $A = A_1 \cup \{k\}$, where A_1 is a (possibly empty) subset of $\{1, 2, \ldots, k-1\}$. Since there are exactly 2^{k-1} such sets A_1, it follows that $N(k) = 2^{k-1}$ for each k. Hence

$$f(n) = \sum_{k=1}^{n} k2^{k-1} = (n-1)2^n + 1,$$

by the result of the previous problem.

Problem 4.

Let $f(x)$ be a polynomial of degree n such that $f(k) = k/(k+1)$ for $k = 0, 1, \ldots, n$. Find $f(n+1)$.

Solution.

Set $g(x) = (x+1)f(x) - x$. Then $g(x)$ is a polynomial of degree $n+1$ which has roots at each of the $n+1$ numbers $0, 1, \ldots, n$. Hence $g(x)$ must be of the form

$$(*) \qquad g(x) = c \prod_{k=0}^{n} (x - k)$$

for some constant c. Setting $x = -1$ in the definition of $g(x)$ we obtain $g(-1) = 1$. By $(*)$ it follows that

$$1 = g(-1) = c \prod_{k=0}^{n} (-1-k) = c(-1)^{n+1}(n+1)!.$$

Hence $c = (-1)^{n+1}/(n+1)!$. Substituting this value into $(*)$ and setting $x = n+1$, we obtain

$$g(n+1) = c \prod_{k=0}^{n} (n+1-k) = \frac{(-1)^{n+1}}{(n+1)!}(n+1)! = (-1)^{n+1},$$

which implies

$$f(n+1) = \frac{g(n+1) + n + 1}{n+2} = \frac{n+1+(-1)^{n+1}}{n+2}.$$

Problem 5.

Let $x_1, x_2, \ldots, x_n$ be n real numbers satisfying

$$\sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} |x_k| = 1.$$

Prove that

$$\left| \sum_{k=1}^{n} \frac{x_k}{k} \right| \leq \frac{1}{2} - \frac{1}{2n}.$$

Solution.

The given conditions on x_k imply that, for any real number λ,

$$\left|\sum_{k=1}^{n} \frac{x_k}{k}\right| = \left|\sum_{k=1}^{n} x_k \left(\frac{1}{k} - \lambda\right)\right|$$
$$\leq \sum_{k=1}^{n} |x_k| \max_{k=1}^{n} \left|\frac{1}{k} - \lambda\right| = \max_{k=1}^{n} \left|\frac{1}{k} - \lambda\right|.$$

Taking $\lambda = 1/2 + 1/2n$, the maximum value of $|1/k - (1/2 + 1/2n)|$ is attained at $k = 1$ and $k = n$ and equal to $1/2 - 1/2n$. The desired inequality then follows.

Problem 6.

Suppose $n = a_1 a_2 \ldots a_{1998}$ is the decimal representation of an integer n consisting of exactly 1998 non-zero digits $a_i \in \{1, 2, \ldots, 9\}$. Show that n is either divisible by 1998, or can be changed to an integer that is divisible by 1998 by replacing some, but not all, of the digits a_i by 0.

Solution.

Let $n_0 = 0$ and for $k = 1, 2, \ldots, 1998$ let n_k denote the number obtained by replacing all but the first k digits of n by the digit 0, i.e., $n_k = a_1 a_2 \ldots a_k 00 \ldots 0$. By the pigeon hole principle, two of the 1999 numbers $n_0, n_1, \ldots, n_{1998}$, say n_{k_1} and n_{k_2} with $k_2 > k_1$ must be congruent modulo 1998. It follows that the difference $n_{k_2} - n_{k_1}$ is divisible by 1998. Now $n_{k_2} - n_{k_1}$ is the number obtained from n by replacing the first k_1 and the last $(1998 - k_2)$ digits by 0. Since $k_2 > k_1$, we have $k_1 + (1998 - k_2) < 1998$, so not all of the 1998 digits are replaced by zero. Thus, the number $n_{k_2} - n_{k_1}$ has the required properties.