1998 UIUC Undergraduate Math Contest

Problem 1.

A sequence $a_0, a_1, a_2 \dots$ of real numbers is defined recursively by

$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad (n = 0, 1, 2, \dots).$$

Find a general formula for a_n.

Problem 2.

Evaluate $\sum_{k=1}^{n} k2^{k-1}$ for $n = 1, 2, \dots$

Problem 3.

Given a nonempty finite set A of real numbers, let $m(A)$ denote the maximal element of A. For $n = 1, 2, \dots$, let $f(n)$ be the sum of $m(A)$, where A runs over the $2^n - 1$ non-empty subsets of the set $\{1, 2, \dots, n\}$. Give an explicit formula for $f(n)$.

Problem 4.

Let $f(x)$ be a polynomial of degree n such that $f(k) = k/(k+1)$ for $k = 0, 1, \dots, n$. Find $f(n+1)$.

Problem 5.

Let $x_1, x_2, \dots, x_n$ be n real numbers satisfying

$$\sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} |x_k| = 1.$$

Prove that

$$\left| \sum_{k=1}^{n} \frac{x_k}{k} \right| \leq \frac{1}{2} - \frac{1}{2n}.$$

Problem 6.

Suppose $n = a_1 a_2 \dots a_{1998}$ is the decimal representation of an integer n consisting of exactly 1998 non-zero digits $a_i \in \{1, 2, \dots, 9\}$. Show that n is either divisible by 1998, or can be changed to an integer that is divisible by 1998 by replacing some, but not all, of the digits a_i by 0.