1997 UIUC UNDERGRAD MATH CONTEST

SOLUTIONS

Problem 1. Let abc represent a three digit number in base 10, with $a \geq c+2$. Let $abc - cba = efg$. Evaluate $efg + gfe$, for all a, b, c, as above.

Solution. Let n_1 denote the number abc, n_2 the number $efg = abc - cba$, and n_3 the number $efg + gfe$. Then $n_1 = 100a + 10b + c$ and $n_2 = 100(a-c) - (a-c) = 100(a-c-1) + 10 \cdot 9 + (10 - a + c)$, so that $e = a - c - 1$, $f = 9$, and $g = 10 - a + c$. (The given conditions on a, b, c ensure that e, f, g fall in the interval $[0, 9]$.) It follows that $n_3 = 101e + 20f + 101g = 101 \cdot 9 + 20 \cdot 9 = 1089$.

Problem 2. Each point in the plane is colored either orange or blue. Prove that one of these colors contains, for each positive value of d, a pair of points at distance d.

Solution. Suppose not. Then there exist positive numbers a and b such that no pair of orange points has distance a and no pair of blue points has distance b. Without loss of generality we may assume $a \leq b$. Now consider a blue point P; such a point has to exist, by our assumption that there are no two orange points at distance a from each other. Since no two blue points have distance b from each other, every point on the circle of radius b around P must be colored orange. Since $a \leq b$, there exist two points on this circle having distance a from each other. These two points are orange points at distance a, which contradicts our assumption. Thus, one of the colors must contain, for every positive distance d, a pair of points at distance d.

Problem 3. Mr. Wisenheimer evaluates on his calculator the expression $\frac{a}{b} - \frac{c}{d}$, where a, b, c, d are positive integers, each less than 1000. The calculator which is known to be accurate to within 10^{-11} for each arithmetic operation, gives the result 0.42857142857. Is Mr. Wisenheimer justified in reporting the answer as **exactly** 3/7? Explain.

Solution. Let $x = 0.42857142857$ be the decimal number displayed by the calculator, and let $\frac{r}{s} = \frac{a}{b} - \frac{c}{d}$ denote the number to be calculated, written as a reduced fraction. It is easy to check that the 11 digits of x after the decimal period represent the first 11 digits in the decimal representation of 3/7, so that x differs from 3/7 by at most 10^{-11}. On the other hand, since the calculator has an error of at most 10^{-11} for each arithmetic operation and the computation of $(r/s) = (a/b) - (c/d)$ involves three arithmetic operations, we know that x also differs from 3/7 by at most $3 \cdot 10^{-11}$. Hence (*) $|(3/7) - (r/s)| \leq 4 \cdot 10^{-11}$. Since b and d are positive integers, each less than 10^3, the denominator s in r/s is at most 10^6. Hence $(3/7) - (r/s) = (3s - 7r)/(7s)$ has denominator at most $7 \cdot 10^6$ and therefore is either 0 or at least $1/(7 \cdot 10^6)$. However, by (*) the second case is impossible, so r/s must be exactly equal to 3/7. Thus, Mr. Wisenheimer is correct in claiming that the result of his calculation is exactly 3/7.

Problem 4. Let $x_1 = x_2 = 1$, and $x_{n+1} = 1996x_n + 1997x_{n-1}$ for $n \geq 2$. Find (with proof) the remainder of x_{1997} upon division by 3.

Solution. First, an easy calculation (which is most conveniently done using congruences modulo 3, though one can do the problem without the use of congruences) shows that $x_1, \ldots, x_6$ have remainders $1, 1, 0, 2, 2, 0$, respectively, when divided by 3. Next, iterating the recurrence relation $x_n = 1996x_{n-1} + 1997x_{n-2}$ four times, one sees that for any $n \geq 6$, the remainder of x_n upon division by 3 is the same as that of x_{n-6}. (With congruences, this calculation amounts to $x_n \equiv x_{n-1} + 2x_{n-2} \equiv 3x_{n-2} + 2x_{n-3} \equiv 2x_{n-3} \equiv 4x_{n-6} \equiv x_{n-6}$.) By induction, it follows that the remainder of x_n upon division by 3 is equal to that of x_r, where r is the remainder of n upon division by 6. Since 1997 has remainder 5 when divided by 6, and x_5 has remainder 2, x_{1997} has remainder 2 when divided by 3.

Problem 5. Let f be a convex function with two continuous derivatives on $[0, 2\pi]$. Show that the integral $\int_0^{2\pi} f(x) \cos x dx$ is positive.

Solution. Since f is convex, $f''(x)$ is positive on $[0, 2\pi]$. Integrating by parts twice, we obtain

$$\begin{aligned}
\int_0^{2\pi} f(x) \cos x &= (\sin x) f(x) \Big|_0^{2\pi} - \int_0^{2\pi} f'(x) \sin x dx \\
&= (\cos x) f'(x) \Big|_0^{2\pi} - \int_0^{2\pi} f''(x) \cos x dx \\
&= f'(2\pi) - f'(0) - \int_0^{2\pi} f''(x) \cos x dx = \int_0^{2\pi} f''(x)(1 - \cos x) dx.
\end{aligned}$$

The last integral is positive since the integrand $f''(x)(1 - \cos x)$ is strictly positive for $0 < x < 2\pi$.

Problem 6. Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \frac{x_n + n x_{n-1}}{n+1}$ for $n \geq 1$. Show that the sequence $\{x_n\}$ converges and find its limit.

Solution. Setting $d_n = x_{n+1} - x_n$, the recurrence relation for x_n translates into $d_n = -\frac{n}{n+1} d_{n-1}$ for $n \geq 1$. Iterating this identity gives

$$d_n = (-1)^n \frac{n}{n+1} \cdot \frac{n-1}{n} \cdots \frac{2}{3} \cdot \frac{1}{2} d_0 = \frac{(-1)^n}{n+1} d_0.$$

Hence

$$x_n = x_0 + \sum_{k=0}^{n-1} d_k = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1},$$

since $x_0 = 0$ and $d_0 = x_1 - x_0 = 1$. As an alternating series with decreasing terms, the series $\sum_{k=0}^{\infty} (-1)^k/(k+1)$ is convergent with sum $\ln(1+1) = \ln 2$, by the Taylor series expansion for $\ln(1+x)$. Hence the sequence $\{x_n\}$ converges with limit $\ln 2$.