1997 UIUC UNDERGRAD MATH CONTEST

Problem 1. Let abc represent a three digit number in base 10, with $a \geq c+2$. Let $abc - cba = efg$. Evaluate $efg + gfe$, for all a, b, c, as above.

Problem 2. Each point in the plane is colored either orange or blue. Prove that one of these colors contains, for each positive value of d, a pair of points at distance d.

Problem 3. Mr. Wisenheimer evaluates on his calculator the expression $\frac{a}{b} - \frac{c}{d}$, where a, b, c, d are positive integers, each less than 1000. The calculator which is known to be accurate to within 10^{-11} for each arithmetic operation, gives the result 0.42857142857. Is Mr. Wisenheimer justified in reporting the answer as **exactly** 3/7? Explain.

Problem 4. Let $x_1 = x_2 = 1$, and $x_{n+1} = 1996x_n + 1997x_{n-1}$ for $n \geq 2$. Find (with proof) the remainder of x_{1997} upon division by 3.

Problem 5. Let f be a convex function with two continuous derivatives on $[0, 2\pi]$. Show that the integral $\int_0^{2\pi} f(x) \cos x dx$ is positive.

Problem 6. Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \frac{x_n + nx_{n-1}}{n+1}$ for $n \geq 1$. Show that the sequence $\{x_n\}$ converges and find its limit.