1996 UIUC UNDERGRAD MATH CONTEST

Problem 1. Let $a_1 < a_2 < \cdots < a_{43} < a_{44}$ be positive integers not exceeding 125. Prove that among the 43 differences $d_i = a_{i+1} - a_i$ $(i = 1, 2, \ldots, 43)$ some value must occur at least 10 times.

Solution. The sum of the 43 differences d_i is $a_{44} - a_1 \leq 124$. If no value among the d_i's occurred more than 9 times, this sum would be at least $9 \cdot 1 + 9 \cdot 2 + 9 \cdot 3 + 9 \cdot 4 + 7 \cdot 5 = 125$, contradicting the above upper bound. Hence some value must occur at least 10 times.

Problem 2. Suppose f is a real positive continuous function on $\mathbf{R}$ with $\int_{-\infty}^{\infty} f(x)dx = 1$. Let $0 < \alpha < 1$, and suppose $[a, b]$ is an interval of *minimal length* with $\int_a^b f(x)dx = \alpha$. Show that $f(a) = f(b)$.

Solution. Let $0 < \alpha < 1$, and suppose $[a, b]$ is an interval *of minimal length* with $\int_a^b f(x)dx = \alpha$. Show that $f(a) = f(b)$. Let $F(y) = \int_y^{y+b-a} f(x)dx$. By hypothesis, $F(a) = \alpha$, and by the fundamental theorem of calculus we have $F'(a) = f(b) - f(a)$. Thus, it suffices to show that $F'(a) = 0$. Since f is continuous, nonnegative and integrable over $(-\infty, \infty)$, $F(y)$ is continuous, nonnegative and tends to zero as $x \to \pm\infty$. The function $F(y)$ therefore attains a finite maximum value M at some point. If we can show that the maximum M is attained at the point $y = a$ it follows that $F'(a) = 0$ as desired. To prove this, suppose that $F(a)$ is not the maximum value of F. Then there exists a number y_0 such that $F(y_0) > F(a)$. Setting $G(y) = \int_{y_0}^{y} f(x)dx$, we have $G(y_0) = 0$ and $G(y_0 + b - a) = F(y_0) > F(a)$. Since f is continuous, so is G, and the intermediate value theorem implies that, for some $y_1 \in (y_0, y_0 + b - a)$, $G(y_1) = F(a) = \alpha$. Since the interval $[y_0, y_1]$ has length $< (b - a)$ this contradicts the assumption that $[a, b]$ is an interval of minimal length with $\int_a^b f(x)dx = \alpha$. Hence the claim is proved.

Problem 3. Evaluate the infinite product $\prod_{k=1}^{\infty} \cos(x2^{-k})$. (Hint: $\sin 2\alpha = 2 \sin \alpha \cos \alpha$.)

Solution. The infinite product is, by definition, the limit of the partial products $P_n = \prod_{k=1}^{n} \cos(x2^{-k})$, as $n \to \infty$. If $x = 0$ this limit is 1. If $x \neq 0$, then the given identity yields

$$P_n = \prod_{k=1}^{n} \frac{\sin(x2^{-k+1})}{2\sin(x2^{-k})} = \frac{\sin x}{2^n \sin(x2^{-n})}$$

(provided n is large enough so that $x2^{-n}$ is not a multiple of π). Since, by l'Hopital's rule, $\lim_{y\to\infty}(\sin y)/y = 1$, we have $\lim_{n\to\infty} 2^n \sin(x2^{-n}) = x$. Hence the value of the given product is $\lim_{n\to\infty} P_n = (\sin x)/x$ if $x \neq 0$.

Problem 4. Let $S = \{0000000, 0000001, \ldots, 1111111\}$ be the set of all binary sequences of length 7. The **distance** of two elements $s_1, s_2 \in S$ is the number of places in which s_1 and s_2 differ. For example, 0001011 and 1001010 have distance 2, since they differ in positions 1 and 7. Show that if T is a subset of S having more than 16 elements then T contains two elements whose distance is at most 2.

Solution. With each element $t \in T$ we can associate a set $S_t \subset S$ consisting of the element t itself and the 7 elements of S that are obtained by switching exactly one of the digits of t. If the distance between any two elements of T were at least three, then the sets S_t would be disjoint, and we would have ($|A|$ denoting the cardinality of a set A) $|S| \geq \sum_{t\in T} |S_t| = 8|T|$, and therefore $|T| \leq |S|/8 = 2^7/8 = 16$. Thus, if $|T| > 16$, there must be two elements in T whose distance is at most 2.

Problem 5. Let a, b, c be real numbers > 1, and let

$$S = \log_a bc + \log_b ca + \log_c ab,$$

where $\log_b x$ denotes the base b logarithm of x. Find, with proof, the smallest possible value of S.

Solution. Setting $A = \log a$, $B = \log b$, $C = \log c$, we have $A, B, C > 0$ (since $a, b, c > 1$) and the sum S becomes $S = \frac{B}{A} + \frac{C}{A} + \frac{C}{B} + \frac{A}{B} + \frac{A}{C} + \frac{B}{C}$. By the arithmetic-geometric mean inequality, this sum is

$$\geq 6 \left(\sqrt{\frac{B \cdot C \cdot C \cdot A \cdot A \cdot B}{A \cdot A \cdot B \cdot B \cdot C \cdot C}} \right)^{1/6} = 6.$$

Hence $S \geq 6$. The example $A = B = C = 1$ shows that this bound is attained, so 6 is the smallest possible value of S.

Problem 6. Suppose $0 \leq s < 1$, $\alpha, \beta > 0$, and $\lfloor \alpha \rfloor > \lfloor \beta \rfloor$. Let $\psi(\alpha, \beta; s)$ be the least positive integer n such that $\lfloor n\alpha + s \rfloor \neq \lfloor n\beta + s \rfloor$. Find an explicit formula for $\psi(\alpha, \beta; s)$ using the floor and ceiling functions. (The floor function $\lfloor x \rfloor$ denotes greatest integer $\leq x$ and the ceiling function $\lceil x \rceil$ denotes the least integer $\geq x$.)

Solution. Let $a = \lfloor \alpha \rfloor, b = \lfloor \beta \rfloor$, $\delta = \alpha - a$, $\gamma = \beta - b$. Then $a > b$ and $\lfloor n\alpha + s \rfloor = na + \lfloor n\delta + s \rfloor$, $\lfloor n\beta + s \rfloor = nb + \lfloor n\gamma + s \rfloor$, and thus

$$\lfloor n\alpha + s \rfloor - \lfloor n\beta + s \rfloor = n(a - b) + \lfloor n\delta + s \rfloor - \lfloor n\gamma + s \rfloor.$$

Since $a > b$ and $\lfloor n\gamma + s \rfloor \leq \lfloor n + s \rfloor \leq n$, the right-hand side is ≥ 0 for every positive integer n, and > 0 if and only if either (i) $a \geq b + 2$ or (ii) $a = b + 1$ and (at least) one of the conditions $n\delta + s \geq 1$ and $n\gamma + s < n$ holds. In the first case we have obviously $\Psi(\alpha, \beta; s) = 1$. In the second case,

$$\begin{aligned}
\psi(\alpha, \beta; s) &= \min\{n \geq 1 : n\delta + s \geq 1 \text{ or } n\gamma + s < n\} \\
&= \min\left(\left\lceil \frac{1 - s}{\delta} \right\rceil, \left\lfloor \frac{s}{1 - \gamma} \right\rfloor + 1 \right) \\
&= \min\left(\left\lceil \frac{1 - s}{\alpha - \lfloor \alpha \rfloor} \right\rceil, \left\lfloor \frac{s}{1 - \beta + \lfloor \beta \rfloor} \right\rfloor + 1 \right).
\end{aligned}$$