1996 UIUC UNDERGRAD MATH CONTEST

Problem 1. Let $a_1 < a_2 < \cdots < a_{43} < a_{44}$ be positive integers not exceeding 125. Prove that among the 43 differences $d_i = a_{i+1} - a_i$ $(i = 1, 2, \ldots, 43)$ some value must occur at least 10 times.

Problem 2. Suppose f is a real positive continuous function on $\mathbf{R}$ with $\int_{-\infty}^{\infty} f(x)dx = 1$. Let $0 < \alpha < 1$, and suppose $[a, b]$ is an interval of *minimal length* with $\int_a^b f(x)dx = \alpha$. Show that $f(a) = f(b)$.

Problem 3. Evaluate the infinite product $\prod_{k=1}^{\infty} \cos(x2^{-k})$. (Hint: $\sin 2\alpha = 2\sin\alpha\cos\alpha$.)

Problem 4. Let $S = \{0000000, 0000001, \ldots, 1111111\}$ be the set of all binary sequences of length 7. The **distance** of two elements $s_1, s_2 \in S$ is the number of places in which s_1 and s_2 differ. For example, 0001011 and 1001010 have distance 2, since they differ in positions 1 and 7. Show that if T is a subset of S having more than 16 elements then T contains two elements whose distance is at most 2.

Problem 5. Let a, b, c be real numbers > 1, and let

$$S = \log_a bc + \log_b ca + \log_c ab,$$

where $\log_b x$ denotes the base b logarithm of x. Find, with proof, the smallest possible value of S.

Problem 6. Suppose $0 \le s < 1$, $\alpha, \beta > 0$, and $\lfloor \alpha \rfloor > \lfloor \beta \rfloor$. Let $\psi(\alpha, \beta; s)$ be the least positive integer n such that $\lfloor n\alpha + s \rfloor \ne \lfloor n\beta + s \rfloor$. Find an explicit formula for $\psi(\alpha, \beta; s)$ using the floor and ceiling functions. (The floor function $\lfloor x \rfloor$ denotes greatest integer $\le x$ and the ceiling function $\lceil x \rceil$ denotes the least integer $\ge x$.)