1996 UIUC UNDERGRAD MATH CONTEST

**Problem 1.** Let \( a_1 < a_2 < \cdots < a_{43} < a_{44} \) be positive integers not exceeding 125. Prove that among the 43 differences \( d_i = a_{i+1} - a_i \) \( (i = 1, 2, \ldots, 43) \) some value must occur at least 10 times.

**Problem 2.** Suppose \( f \) is a real positive continuous function on \( \mathbb{R} \) with \( \int_{-\infty}^{\infty} f(x)dx = 1 \). Let \( 0 < \alpha < 1 \), and suppose \( [a, b] \) is an interval of minimal length with \( \int_a^b f(x)dx = \alpha \). Show that \( f(a) = f(b) \).

**Problem 3.** Evaluate the infinite product \( \prod_{k=1}^{\infty} \cos(x2^{-k}) \). (Hint: \( \sin 2\alpha = 2 \sin \alpha \cos \alpha \).)

**Problem 4.** Let \( S = \{0000000, 0000001, \ldots, 1111111\} \) be the set of all binary sequences of length 7. The **distance** of two elements \( s_1, s_2 \in S \) is the number of places in which \( s_1 \) and \( s_2 \) differ. For example, 0001011 and 1001010 have distance 2, since they differ in positions 1 and 7. Show that if \( T \) is a subset of \( S \) having more than 16 elements then \( T \) contains two elements whose distance is at most 2.

**Problem 5.** Let \( a, b, c \) be real numbers \( > 1 \), and let

\[
S = \log_a bc + \log_b ca + \log_c ab,
\]

where \( \log_b x \) denotes the base \( b \) logarithm of \( x \). Find, with proof, the smallest possible value of \( S \).

**Problem 6.** Suppose \( 0 \leq s < 1, \alpha, \beta > 0 \), and \( \lfloor \alpha \rfloor > \lfloor \beta \rfloor \). Let \( \psi(\alpha, \beta; s) \) be the least positive integer \( n \) such that \( \lfloor n\alpha + s \rfloor \neq \lfloor n\beta + s \rfloor \). Find an explicit formula for \( \psi(\alpha, \beta; s) \) using the floor and ceiling functions. (The floor function \( \lfloor x \rfloor \) denotes greatest integer \( \leq x \) and the ceiling function \( \lceil x \rceil \) denotes the least integer \( \geq x \).)