U OF I UNDERGRADUATE MATH CONTEST
March 8, 2008
Solutions

1. *Does there exist a multiple of* 2008 *whose decimal representation involves only a single digit (such as* 11111 *or* 22222222*)?*

   Solution. The answer is yes; specifically, we will show that there exists a multiple of 2008 of the form $888\dots 8$.

   Given a digit $d \in \{1, 2, \dots, 9\}$, let $N_{d,k}$ be the number whose decimal representation consists of k digits d. Note that

$$N_{d,k} = d\sum_{i=0}^{k-1} 10^i = \frac{d(10^k - 1)}{9}.$$

   Thus, a given positive integer m has a multiple of this form if and only if the congruence (1) $d(10^k - 1) \equiv 0 \bmod 9m$ has a solution k. We apply this with $d = 8$ and $m = 2008$. Then (1) is equivalent to (2) $10^k - 1 \equiv 0 \bmod 9(2008/8) = 9 \cdot 251$. Since $10^k \equiv 1^k = 1 \bmod 9$ for any positive integer k, (2) is equivalent to (3) $10^k \equiv 1 \bmod 251$. Now, 251 is prime, so by Fermat's Theorem, we have $10^{251-1} \equiv 1 \bmod 251$. Thus, (3) holds for $k = 250$, and so the number $N_{8,250} = \underbrace{88\dots 8}_{250}$ is divisible by 2008.

2. *What is the maximal value of the integral* $\int_0^1 f(x)x^{2008}dx$ *among all nonnegative continuous functions* f *on the interval* $[0, 1]$ *satisfying* $\int_0^1 f(x)^2 dx = 1$*?*

   Solution. Let $I = \int_0^1 f(x)x^{2008}dx$ be the integral whose maximum we seek. Applying the integral version of Cauchy-Schwarz with the functions $f(x)$ and x^{2008}, we get, for any f with $\int_0^1 f(x)^2 dx = 1$,

$$I^2 \le \int_0^1 f(x)^2 dx \int_0^1 x^{4016} dx = \frac{1}{4017}.$$

   Thus, $I \le 1/\sqrt{4017}$. Moreover, this upper bound is achieved by taking $f(x) = cx^{2008}$, with $c = \sqrt{4017}$ (so that f satisfies $\int_0^1 f(x)^2 dx = 1$). Thus, $1/\sqrt{4017}$ is the maximal value for I.

3. *Find, with proof, all functions* $f : \mathbf{R} \to \mathbf{R}$ *satisfying*

$$|f(x + y) - f(x - y) - y| \le y^2$$

   for all $x, y \in \mathbf{R}$.

Solution. Any function of the form $f_c(x) = x/2 + c$, where c is a constant, satisfies $f_c(x+y) - f_c(x-y) - y \equiv 0$ for all x and y, and hence trivially satisfies the above inequality. We will show that these are the only such functions.

Suppose $f(x)$ is solution to the given inequality. Set $g(x) = f(x) - x/2$. Then

$$\begin{aligned}|g(x+y) - g(x-y)| &= |f(x+y) - (x+y)/2 - f(x-y) + (x-y)/2 - y| \\ &= |f(x+y) - f(x-y) - y| \le y^2\end{aligned}$$

for all $x, y \in \mathbf{R}$. Dividing by y and letting $y \to 0$, we conclude

$$\lim_{y\to 0} \frac{g(x+y) - g(x-y)}{y} = \lim_{y \to 0} y = 0,$$

for all $x \in \mathbf{R}$. Thus g is a differentiable function, with derivative equal to 0 everywhere. It follows that g must be a constant function, say, $g(x) = c$ for all x. Therefore $f(x) = x/2 + g(x) = x/2 + c$, as claimed.

4. *Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and for $n \ge 4$ define a_n to be the last digit of the sum of the preceding* **three** *terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 124734419447.... Determine, with proof, whether or not the string 1001 occurs in this sequence. (Hint: Do* **not** *attempt this by brute force!)*

   Solution. First note that the sequence can be continued backwards in a unique manner by setting $a_{n-1} = a_{n+2} - a_{n+1} - a_n \mod 10$. Doing so, one finds that the first four terms prior to the given terms are 1, 0, 0, and 1. Thus, the string 1001 occurs in the extended sequence. To show that it also occurs in the original sequence (i.e., to the right of 1247...), note that the sequence is uniquely determined, backwards and forwards, by any three consecutive digits in the sequence. Since there are finitely many possibilities for such triples of consecutive digits, one such triple must occur again in the sequence, and the sequence is therefore is periodic (in both directions). In particular, any string that occurs somewhere in the extended sequence, occurs infinitely often and arbitrarily far out along the given sequence. Hence 1001 does occur infinitely often in this sequence. (While this term occurs immediately to the left of the given initial string 1247..., its first occurrence to the right is at the 120-th term. This would be hard to discover by a hand calculation!)

5. *Let n be a positive integer, and denote by S_n the set of all permutations of $\{1, 2, \dots, n\}$. Given a permutation $\sigma \in S_n$, define its* ***perturbation index*** *$P(\sigma)$ as*

$$P(\sigma) = \#\{k \in \{1, \dots, n\} : \sigma(k) \ne k\};$$

   i.e., $P(\sigma)$ denotes the number of elements in $\{1, \dots, n\}$ that are "perturbed" by σ, in the sense of being mapped to a different element. Find the average perturbation index of a permutation in S_n, i.e.,

$$\frac{1}{\#S_n} \sum_{\sigma \in S_n} P(\sigma).$$

Solution. We have

$$\begin{aligned}
\sum_{\sigma\in S_n} P(\sigma) &= \sum_{\sigma\in S_n} \sum_{\substack{k=1\\ \sigma(k)\neq k}}^{n} 1\\
&= \sum_{k=1}^{n} \sum_{\substack{\sigma\in S_n\\ \sigma(k)\neq k}} 1\\
&= \sum_{k=1}^{n} \#\{\sigma\in S_n : \sigma(k)\neq k\}\\
&= \sum_{k=1}^{n} (n! - \#\{\sigma\in S_n : \sigma(k) = k\})\\
&= \sum_{k=1}^{n} (n! - (n-1)!)\\
&= (n-1)n!,
\end{aligned}$$

since, for any k, there are exactly $(n-1)!$ permutations in S_n that fix k. Since $\#S_n = n!$, it follows that the average perturbation index is $n-1$.

6. *Let $\mathcal{A}$ be a collection of 100 distinct, nonempty subsets of the set $\{0,1,\dots,9\}$. Show that there exist two (distinct) sets $A, A' \in \mathcal{A}$ whose symmetric difference has at most two elements. (The symmetric difference of two sets A and A' is defined as the set of elements that are in one of the two sets, but not in both, i.e., the set $(A\cup A')\setminus(A\cap A')$.)*

Solution. Let $A\Delta B$ denote the symmetric difference of two sets A and B, and let $d(A,B) = |A\Delta B|$ denote the number of elements in $A\Delta B$. It is easy to see that the function d satisfies the triangle inequality:

$$d(A,C) \le d(A,B) + d(B,C). \tag{1}$$

Now, let $\mathcal{A}$ be a collection of subsets of $\{0,1,\dots,9\}$ with $|\mathcal{A}| = 100$. In the above terminology, we need to show that if $\mathcal{A}$ has at least 100 elements then it contains two elements, A and A', such that $d(A,A') \le 2$.

Given $A \in \mathcal{A}$, define a "neighborhood" of A by

$$\mathcal{U}(\mathcal{A}) = \{B \subset \{0,1,\dots,9\} : B \neq \emptyset, d(B,A) \le 1\};$$

i.e., $\mathcal{U}(\mathcal{A})$ consists all nonempty subsets of $\{0,1,\dots,9\}$ whose symmetric difference with A has at most 1 element. We are going to estimate the sum of the cardinalities of these "neighborhoods",

$$S = \sum_{A\in\mathcal{A}} |\mathcal{U}(\mathcal{A})|.$$

To this end, note that $\mathcal{U}(\mathcal{A})$ consists of the following sets: (i) the set A itself, (ii) any proper nonempty subset of A obtained by removing exactly one element from A, and (iii) any set obtained by adding to A exactly one element from $\{0, 1, \ldots, 9\}$.

If A has k elements with $k \geq 2$, then there are exactly k sets of type type (ii), and $(10 - k)$ sets of type (iii), so $\mathcal{U}(\mathcal{A})$ has exactly $1 + k + (10 - k) = 11$ elements. If A has 1 element, then there is no set of type (ii) (since removing the single element of A would leave an empty set), so in that case $\mathcal{U}(\mathcal{A})$ has 10 elements. Setting

$$\mathcal{A}_1 := \{A \in \mathcal{A} : |A| = 1\}, \quad \mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$$

we therefore have

$$\begin{aligned} S &= 10|\mathcal{A}_1| + 11|\mathcal{A}_2| \\ &= 11|\mathcal{A}| - |\mathcal{A}_1| \\ &\geq 11 \cdot 100 - 10 = 1090 \end{aligned} \tag{2}$$

by our assumption $|\mathcal{A}| = 100$ and the trivial bound $|\mathcal{A}_1| \leq 10$, since there are 10 one element subsets of $\{0, 1, \ldots, 9\}$.

On the other hand, if the sets $\mathcal{U}(\mathcal{A})$, $A \in \mathcal{A}$, were pairwise disjoint, we would have

$$\begin{aligned} S &= \left| \bigcup_{A \in \mathcal{A}} \mathcal{U}(\mathcal{A}) \right| \\ &\leq |\{B \subset \{0, 1, \ldots, 9\} : B \neq \emptyset\}| \\ &= 2^{10} - 1 = 1023, \end{aligned} \tag{3}$$

contradicting the lower bound (2). Thus, two of these sets, say $\mathcal{U}(\mathcal{A})$ and $\mathcal{U}(\mathcal{A}')$, must have a nonempty intersection. By the definition of the neighborhoods $\mathcal{U}$ this means that there exists a nonempty subset $B \subset \{0, 1, \ldots, 9\}$ such that $d(B, A) \leq 1$ and $d(B, A') \leq 1$. By the triangle inequality (1), this implies

$$d(A, A') \leq d(A, B) + d(B, A') \leq 1 + 1 \leq 2.$$

which is what we wanted to show.