U OF I UNDERGRADUATE MATH CONTEST
March 8, 2008

1. *Does there exist a multiple of* 2008 *whose decimal representation involves only a single digit (such as* 11111 *or* 22222222*)?*

2. *What is the maximal value of the integral* $\int_0^1 f(x)x^{2008}dx$ *among all nonnegative continuous functions* f *on the interval* $[0,1]$ *satisfying* $\int_0^1 f(x)^2 dx = 1$*?*

3. *Find, with proof, all functions* $f : \mathbf{R} \to \mathbf{R}$ *satisfying*

$$|f(x+y) - f(x-y) - y| \leq y^2$$

   for all $x, y \in \mathbf{R}$.

4. *Let* $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, *and for* $n \geq 4$ *define* a_n *to be the last digit of the sum of the preceding* **three** *terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form)* $124734419447\ldots$. *Determine, with proof, whether or not the string* 1001 *occurs in this sequence. (Hint: Do* ***not*** *attempt this by brute force!)*

5. *Let* n *be a positive integer, and denote by* S_n *the set of all permutations of* $\{1,2,\ldots,n\}$. *Given a permutation* $\sigma \in S_n$, *define its* ***perturbation index*** $P(\sigma)$ *as*

$$P(\sigma) = \#\{k \in \{1,\ldots,n\} : \sigma(k) \neq k\};$$

   i.e., $P(\sigma)$ *denotes the number of elements in* $\{1,\ldots,n\}$ *that are "perturbed" by* σ, *in the sense of being mapped to a different element. Find the average perturbation index of a permutation in* S_n, *i.e.,*

$$\frac{1}{\#S_n} \sum_{\sigma \in S_n} P(\sigma).$$

6. *Let* $\mathcal{A}$ *be a collection of* 100 *distinct, nonempty subsets of the set* $\{0,1,\ldots,9\}$. *Show that there exist two (distinct) sets* $A, A' \in \mathcal{A}$ *whose symmetric difference has at most two elements. (The symmetric difference of two sets* A *and* A' *is defined as the set of elements that are in one of the two sets, but not in both, i.e., the set* $(A \cup A') \setminus (A \cap A')$*.)*

[Solutions at `http://www.math.uiuc.edu/contests.html`]