U OF I UNDERGRADUATE MATH CONTEST
April 14, 2007
Solutions

1. *Let*
$$f(n) = (1^2 + 1)1! + (2^2 + 1)2! + \cdots + (n^2 + 1)n!.$$
Find a simple general formula for $f(n)$*.*

Solution. We will show by induction that

$$(*) \qquad f(n) = n(n+1)!$$

for $n = 1, 2, \ldots$. For $n = 1$, $(*)$ holds trivially. Let now $n \geq 1$, and assume that $(*)$ holds for this value of n. Then

$$\begin{aligned} f(n+1) &= f(n) + ((n+1)^2 + 1)(n+1)! \\ &= n(n+1)! + (n^2 + 2n + 2)(n+1)! \\ &= (n^2 + 3n + 2)(n+1)! \\ &= (n+1)(n+2)(n+1)! = (n+1)(n+2)!, \end{aligned}$$

which proves $(*)$ with $n+1$ in place of n, completing the induction.

2. *Prove that for every odd integer* n *the sum* $1^n + 2^n + \cdots + n^n$ *is divisible by* n^2*.*

Solution. For $n = 1$, the assertion is trivially true. Suppose therefore that n is greater than 1 and odd. Then

$$1^n + 2^n + \cdots + n^n = \sum_{k=1}^{(n-1)/2} (k^n + (n-k)^n) + n^n$$
$$= \sum_{k=1}^{(n-1)/2} \left(k^n + n^n + \binom{n}{1} n^{n-1}(-k)^1 + \cdots + \binom{n}{n-1} n^1 (-k)^{n-1} + (-k)^n \right) + n^n.$$

Since n is odd, the terms k^n and $(-k)^n$ cancel each other out. Of the remaining terms, all divisible by n^2, so the assertion follows.

3. *For any positive integer* k *let* $f_1(k)$ *denote the sum of the squares of the digits of* k *(when written in decimal), and for* $n \geq 2$ *define* $f_n(k)$ *iteratively by* $f_n(k) = f_1(f_{n-1}(k))$*. Find* $f_{2007}(2006)$*.*

Solution. Starting from $k = 2006$ and iterating the map "sum of squares of the digits" we obtain the chain $2006 \to 40 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4 \to 16$, after which the sequence repeats itself, with period 8. Thus, $f_1(2006) = 40$, $f_2(2006) = 16$, etc., and $f_{n+8}(2006) = f_n(2006)$ for all integers $n \geq 1$. Since $2007 = 8 \cdot 250 + 7$, it follows that follows that $f_{2007}(2006) = f_7(2006) = 42$.

4. *Determine, with proof, whether the series*

$$\sum_{n=1}^{\infty}\left(e-\left(1+\frac{1}{n}\right)^n\right)$$

converges.

Solution. We claim that the series diverges. To show this, we first derive a bound for $\ln(1+x)$, using the Taylor expansion $\ln(1+x)=\sum_{n=1}^{\infty}(-x)^{n+1}/n$. For $0<x<1$ the latter series is an alternating series with decreasing terms, so the successive partial sums of this series alternately undershoot and overshoot the limit, $\ln(1+x)$. In particular, we have, for $0<x<1$,

$$\ln(1+x)\le x-\frac{x^2}{2}+\frac{x^3}{3}\le x-\frac{x^2}{2}+\frac{x^2}{3}=x-\frac{x^2}{6}.$$

It follows that, for $n\ge 2$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \exp\left\{n\ln\left(1+\frac{1}{n}\right)\right)\}\\ &\le \exp\left\{n\left(\frac{1}{n}-\frac{1}{6n^2}\right)\right\}=e^{1-1/(6n)}.\end{aligned}$$

Hence, if a_n denotes the n-th term of the given series, we have the lower bound

$$a_n=e-\left(1+\frac{1}{n}\right)^n\ge e-e^{1-1/(6n)}=e\left(1-e^{-1/(6n)}\right)$$

By another application of the alternating series properties, we see that, for $0\le x\le 1$,

$$e^{-x}=\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\le 1-x+\frac{x^2}{2}\le 1-\frac{x}{2}.$$

Combining this with the above inequality for a_n gives

$$a_n\ge e\left(1-e^{-1/(6n)}\right)\ge e\left(1-\left(1-\frac{3}{n}\right)\right)=\frac{e}{12n},$$

for $n\ge 2$. Since the harmonic series $\sum_{n=1}^{\infty}1/n$ diverges, it follows that the series over a_n diverges as well.

Alternate solution (due to Ben Kaduk). By the binomial theorem, we have for $n\ge 2$

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}\\ &= 1+\sum_{k=1}^{n}\frac{1}{k!}\frac{n(n-1)\dots(n-k+1)}{n^k}\\ &\le 1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\sum_{k=3}^{n}\frac{1}{k!}\\ &\le \sum_{k=0}^{\infty}\frac{1}{k!}-\frac{1}{2n}=e-\frac{1}{2n}.\end{aligned}$$

Hence $a_n \geq 1/(2n)$ for $n \geq 2$, and comparison with the harmonic series yields the divergence of $\sum_{n=1}^{\infty} a_n$.

5. *Suppose $P_1, \dots, P_{12}$ are points on the unit circle $x^2 + y^2 = 1$, and let*

$$S = S(P_1, \dots, P_{12}) = \sum_{1 \leq i < j \leq 12} |P_i P_j|^2,$$

where $|P_i P_j|$ denotes the distance between P_i and P_j. In other words, S is the sum of the squares of the pairwise distances between the points $P_1, \dots, P_{12}$. Determine, with proof, the largest possible value of S among all choices of the points $P_1, \dots, P_{12}$ on the unit circle.

Solution. We represent the points P_i by complex numbers z_i. Points on the unit circle correspond to complex numbers of absolute value 1, and the distance between two such points is the absolute value of the difference between the corresponding complex numbers. We thus can write

$$S = S(z_1, \dots, z_{12}) = \sum_{1 \leq i < j \leq 12} |z_i - z_j|^2$$

and the problem reduces to that of maximizing this function subject to the condition that $|z_i| = 1$ for all i.

Since $|z_i - z_i| = 0$ and $|z_i - z_j| = |z_j - z_i|$, if we extend the above sum over $1 \leq i < j \leq 12$ to *all* pairs of indices (i, j), with $1 \leq i, j \leq 12$, we count each summand twice. Thus, S is exactly equal to half this extended sum, i.e.,

$$S = \frac{1}{2} \sum_{i=1}^{12} \sum_{j=1}^{12} |z_i - z_j|^2$$

Using the assumption that $|z_i| = 1$, we can expand the summands into

$$\begin{aligned} |z_i - z_j|^2 &= (z_i - z_j)(\overline{z_i} - \overline{z_j}) \\ &= |z_i|^2 + |z_j|^2 - z_i \overline{z_j} - \overline{z_i} z_j = 2 - z_i \overline{z_j} - \overline{z_i} z_j. \end{aligned}$$

Substituting this into the above identity for S, we get

$$\begin{aligned} S &= \frac{1}{2} \sum_{i=1}^{12} \sum_{j=1}^{12} (2 - z_i \overline{z_j} - \overline{z_i} z_j) \\ &= \frac{1}{2} \left(2 \cdot 12^2 - 2 \left| \sum_{i=1}^{12} z_i \right|^2 \right) \end{aligned}$$

Thus, $S \leq 12^2 = 144$ for any choice of the numbers z_i (subject to $|z_i| = 1$), and $S = 144$ whenever

$$\sum_{i=1}^{12} z_i = 0.$$

The latter condition can be achieved, for example, by choosing the first 6 points, $z_1, \ldots, z_6$, arbitrarily on the unit circle, and letting the remaining 6 points be the points located diametrically opposite the 6 chosen points, i.e., $z_7 = -z_1, \ldots, z_{12} = -z_6$. Hence 144 is the maximal value of S.

6. *Let a_n $(n = 0, 1, \ldots)$ be a bounded sequence of positive integers that satisfies*

$$a_n \left(a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2\right) = a_{n-1}^3 a_1 + a_{n-2}^3 a_2 + \cdots + a_{n-2007}^3 a_{2007} \quad (n \geq 2007).$$

Show that the sequence eventually becomes periodic.

Solution. Let $\mathbf{a}_n$ denote the 2007-tuple $(a_n, \ldots, a_{n-2006})$. The given recurrence can be rewritten as

$$a_n = \frac{a_{n-1}^3 a_1 + a_{n-2}^3 a_2 + \cdots + a_{n-2007}^3 a_{2007}}{a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2}. \tag{1}$$

Thus the value of $\mathbf{a}_{n-1}$, along with that of the (constant) tuple $\mathbf{a}_{2007} = (a_{2007}, \ldots, a_1)$, completely determines that of a_n, hence, by induction, the values of a_{n+k} and $\mathbf{a}_{n+k}$ for all $k \geq 0$.

Since the numbers a_n are bounded positive integers, there are only finitely many possible values for the tuple $\mathbf{a}_n$. By the pigeonhole principle, it therefore follows that there exist positive integers $n < m$ with $\mathbf{a}_n = \mathbf{a}_m$. In view of the above remark, this implies $a_{n+k} = a_{m+k}$ for all integers $k \geq 0$. Thus, a_n is eventually periodic with period $m - n$.

Note. In the original version of this problem, the assumption that the a_n be *positive* was missing. This assumption ensures that the denominator in (1) is positive, so the given recurrence can be written in the above form. Ben Kaduk constructed a example showing that, i if the a_n are allowed to be 0, the conclusion need not not hold.