1. Let
\[ f(n) = (1^2 + 1)1! + (2^2 + 1)2! + \cdots + (n^2 + 1)n! . \]
Find a simple general formula for \( f(n) \).

2. Prove that for every odd integer \( n \) the sum \( 1^n + 2^n + \cdots + n^n \) is divisible by \( n^2 \).

3. For any positive integer \( k \) let \( f_1(k) \) denote the sum of the squares of the digits of \( k \) (when written in decimal), and for \( n \geq 2 \) define \( f_n(k) \) iteratively by \( f_n(k) = f_1(f_{n-1}(k)) \). Find \( f_{2007}(2006) \).

4. Determine, with proof, whether the series
\[ \sum_{n=1}^{\infty} \left( e - \left( 1 + \frac{1}{n} \right)^n \right) \]
converges.

5. Suppose \( P_1, \ldots, P_{12} \) are points on the unit circle \( x^2 + y^2 = 1 \), and let
\[ S = S(P_1, \ldots, P_{12}) = \sum_{1 \leq i < j \leq 12} |P_iP_j|^2 , \]
where \( |P_iP_j| \) denotes the distance between \( P_i \) and \( P_j \). In other words, \( S \) is the sum of the squares of the pairwise distances between the points \( P_1, \ldots, P_{12} \). Determine, with proof, the largest possible value of \( S \) among all choices of the points \( P_1, \ldots, P_{12} \) on the unit circle.

6. Let \( a_n \) (\( n = 0, 1, \ldots \)) be a bounded sequence of positive integers that satisfies
\[ a_n \left( a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2 \right) = a_{n-1}^3 a_{n-2} + a_{n-2} a_{n-2}^2 + \cdots + a_{n-2007}^3 a_{2007} \quad (n \geq 2007) . \]
Show that the sequence eventually becomes periodic.

[Solutions at http://www.math.uiuc.edu/contests.html]