U OF I UNDERGRADUATE MATH CONTEST
April 14, 2007

1. *Let*
$$f(n) = (1^2+1)1! + (2^2+1)2! + \cdots + (n^2+1)n!.$$
Find a simple general formula for $f(n)$.

2. *Prove that for every odd integer n the sum $1^n + 2^n + \cdots + n^n$ is divisible by n^2.*

3. *For any positive integer k let $f_1(k)$ denote the sum of the squares of the digits of k (when written in decimal), and for $n \geq 2$ define $f_n(k)$ iteratively by $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{2007}(2006)$.*

4. *Determine, with proof, whether the series*
$$\sum_{n=1}^{\infty} \left(e - \left(1 + \frac{1}{n}\right)^n \right)$$
converges.

5. *Suppose $P_1, \ldots, P_{12}$ are points on the unit circle $x^2 + y^2 = 1$, and let*
$$S = S(P_1, \ldots, P_{12}) = \sum_{1 \leq i < j \leq 12} |P_i P_j|^2,$$
where $|P_i P_j|$ denotes the distance between P_i and P_j. In other words, S is the sum of the squares of the pairwise distances between the points $P_1, \ldots, P_{12}$. Determine, with proof, the largest possible value of S among all choices of the points $P_1, \ldots, P_{12}$ on the unit circle.

6. *Let a_n $(n = 0, 1, \ldots)$ be a bounded sequence of positive integers that satisfies*
$$a_n \left(a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2\right) = a_{n-1}^3 a_1 + a_{n-2}^3 a_2 + \cdots + a_{n-2007}^3 a_{2007} \quad (n \geq 2007).$$
Show that the sequence eventually becomes periodic.

[Solutions at `http://www.math.uiuc.edu/contests.html`]