# UIUC UNDERGRADUATE MATH CONTEST
# April 8, 2006
# Solutions

1. *Determine*, without numerical calculations, *which of the two numbers* $\sqrt{2005}^{\sqrt{2006}}$ *and* $\sqrt{2006}^{\sqrt{2005}}$ *is larger.*

   **Solution.** We will show that the first of the two numbers, $\sqrt{2005}^{\sqrt{2006}}$, is the larger one. Taking the $(\sqrt{2005}\cdot\sqrt{2006})$-th root of the two given numbers, this amounts to showing that $\sqrt{2005}^{1/\sqrt{2005}}$ is larger than $\sqrt{2006}^{1/\sqrt{2006}}$. This will follow provided we can show that the function $f(x) = x^{1/x}$ is decreasing in an interval that includes $\sqrt{2005}$ and $\sqrt{2006}$. To do this, we compute the derivative of $f(x)$: Writing $f(x) = \exp\{(\ln x)/x\}$, we have, by the chain rule,

   $$f'(x) = \exp\left\{\frac{\ln x}{x}\right\}\left(\frac{x(1/x) - (\ln x)\cdot 1}{x^2}\right) = x^{1/x}\frac{1-\ln x}{x^2}.$$

   Thus we see that $f'(x)$ is negative, and hence $f(x)$ is decreasing, for $x > e$. Since $\sqrt{2005} > e$, this is what we need, with plenty of room to spare.

2. *Let* $f(x) = e^{x^2}\sin x$. *Find, with proof,* $f^{(2006)}(0)$, *the* 2006*th derivative of* $f$ *at* 0.

   **Solution.** We use the connection between derivatives at 0 and coefficients of Taylor series: if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the Taylor expansion of $f$ at 0, then $a_n = f^{(n)}(0)/n!$. Now, the Taylor series of $f(x) = e^{x^2}\sin x$ is the product of the Taylor series for $e^{x^2}$ and $\sin x$, which are $\sum_{n=0}^{\infty} x^{2n}/n!$ and $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$, respectively. Since the first of these series involves only even powers of $x$, and the second involves only odd powers of $x$, their product contains only odd powers of $x$. Hence, all even-indexed coefficients in the Taylor series for $f(x)$ are 0 and so, in particular, $f^{(2006)}(0) = a_{2006}(2006)! = 0$.

3. *Evaluate the series*

   $$\sum_{n=0}^{\infty}\frac{1}{2006^{2^n} - 2006^{-2^n}} = \frac{1}{2006^1 - 2006^{-1}} + \frac{1}{2006^2 - 2006^{-2}} + \frac{1}{2006^4 - 2006^{-4}} + \cdots$$

   *and express it as a rational number.*

**Solution.** We claim that the given series is equal to 1/2005.

More generally, let

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}, \quad S_N(x) = \sum_{n=0}^{N} \frac{x^{2^n}}{1 - x^{2^{n+1}}}.$$

Note that setting $x = 1/2006$ in $S(x)$ gives the series of the problem. We will show that, for any $x$ with $0 < x < 1$,

$$S(x) = \frac{x}{1-x}, \qquad (*)$$

and so, in particular, $S(1/2006) = (1/2006)(1 - 1/2006) = 1/2005$, as claimed.

We give two proofs for $(*)$.

**First proof of $(*)$.** We use a "telescoping" argument, based on the elementary identity

$$\frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}}.$$

Substituting this into the partial sums $S_N(x)$ (we work with partial sums rather than the infinite series $S(x)$ in order to avoid possible convergence problems), we get

$$\begin{aligned} S_N(x) &= \left(\frac{1}{1-x} - \frac{1}{1-x^2}\right) + \left(\frac{1}{1-x^2} - \frac{1}{1-x^4}\right) \\ &\quad + \cdots + \left(\frac{1}{1-x^{2^N}} - \frac{1}{1-x^{2^{N+1}}}\right) \\ &= \frac{1}{1-x} - \frac{1}{1-x^{2^{N+1}}}. \end{aligned}$$

As $N \to \infty$ here, the last term tends to 1, and we obtain

$$S(x) = \lim_{N \to \infty} S_N(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x},$$

proving $(*)$.

**Second proof of $(*)$.** A completely different proof of $(*)$ goes as follows: First expand each term in $S(x)$ into a geometric series:

$$S(x) = \sum_{n=0}^{\infty} x^{2^n} \sum_{k=0}^{\infty} x^{2^{n+1}k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{2^n(1+2k)}.$$

Note that all terms in this double series are positive, so we can rearrange the terms in this series. Since each positive integer has a unique decomposition as a power of 2 times an odd positive integer, the exponents $2^n(1+2k)$ in the latter double series correspond in a one-to-one manner to the positive integers, and the latter double sum, and hence $S(x)$, is therefore exactly $\sum_{m=1}^{\infty} x^m = x/(1-x)$. This again gives $(*)$.

4. *For any positive integer $n$, define a sequence $\{n_k\}_{k=0}^{\infty}$ as follows: Set $n_0 = n$, and for each $k \geq 1$, let $n_k$ be the sum of the (decimal) digits of $n_{k-1}$. For example, for $n = 1729$ we get the sequence $1729, 19, 10, 1, 1, 1, \ldots$. In general, for any given starting value $n$, the resulting sequence $\{n_k\}$ eventually stabilizes at a single digit value. Let $f(n)$ denote this value; for example, $f(1729) = 1$. Determine $f(2^{2006})$.*

   **Solution.** The key to this problem is the fact (which underlies the well-known disibility test by 9) that the sum of the decimal digits of a number has the same remainder modulo 9 as the number itself. Thus, each of the numbers $n_k$ in the given sequence has the same remainder modulo 9. In particular, $f(n)$ has the same remainder modulo 9 as $n$, and since $f(n)$ must be among $\{1, 2, \ldots, 9\}$, the remainder of $n$ modulo 9 determines $f(n)$ uniquely. Thus, it remains to determine the remainder of $2^{2006}$ upon division by 9. This is easy using congruence calculus: We have $2^6 = 64 \equiv 1 \bmod 9$, and so

$$2^{2006} = 4 \cdot 2^{2004} = 4 \cdot (2^6)^{334} \equiv 4 \cdot 1^{334} = 4 \bmod 9.$$

   Hence $f(2^{2006}) = 4$.

5. *Let $D = \{d_1, d_2, \ldots, d_{10}\}$ be a set of 10 distinct positive integers. Show that any sequence of 2006 integers from $D$ contains a block of one or more consecutive terms whose product is the square of a positive integer.*

   **Solution.** Let a set $D$ and a sequence $\{a_i\}_{i=1}^{2006}$ with $a_i \in D$ be given as in the problem. For $n = 1, 2, \ldots, 2006$ let $P_n = \prod_{i=1}^{n} a_i$ denote the product of the first $n$ terms and set $P_0 = 1$,

   Note that any block of consecutive terms from the sequence $\{a_i\}$ is of the form $\prod_{i=m+1}^{n} a_i = P_n/P_m$ for some integers $m$ and $n$ with $0 \leq m < n \leq 2006$. Thus, the problem amounts to showing that, for a suitable choice of $(m, n)$ with $0 \leq m < n \leq 2006$, $P_n/P_m$ is a perfect square. Since each $a_i$ is among the numbers $d_1, d_2, \ldots, d_{10}$, each $P_n$ is of the form $P_n = \prod_{i=1}^{10} d_i^{\alpha_{in}}$, where the exponents $\alpha_{in}$ are nonnegative integers. (With $\alpha_{i0} = 0$ for $i = 1, 2, \ldots, 10$ this also holds for $P_0$.)

   Note that, by the definition of $P_n$ as the product of the first $n$ terms of the sequence $\{a_i\}$, the exponents $\alpha_{in}$ are non-decreasing in $n$, for each $i$. Thus, for $0 \leq m < n \leq 2006$, $P_n/P_m = \prod_{i=m+1}^{n} a_i = \prod_{i=1}^{n} d_i^{\alpha_{in} - \alpha_{im}}$, where the exponents $\alpha_{in} - \alpha_{im}$ are nonnegative integers. Hence $P_n/P_m$ will certainly be a perfect square if the numbers $\alpha_{kn} - \alpha_{km}$, $k = 1, \ldots, 10$, are all even.

   The latter condition holds if and only if the vectors $\boldsymbol{\alpha}_n = (\alpha_{1n}, \ldots, \alpha_{10n})$ and $\boldsymbol{\alpha}_m = (\alpha_{1m}, \ldots, \alpha_{10m})$ have the same parity in each component. Now, since each $\boldsymbol{\alpha}_n$, $n = 0, 1, 2, \ldots, 2006$, is a vector with 10 components, there are $2^{10} = 1024$ possible parity combinations for these components. Since we have $2007 > 1024$ such vectors, by the pigeonhole principle two of these must have the same parity combination. Denoting the indices of these two vectors by $m$ and $n$ (ordered so that $0 \leq m < n \leq 2006$), we then have that $P_n/P_m$ is a perfect square, as claimed.

6. *Given a real number $\alpha$ with $0 \leq \alpha < 1$, define an $\alpha$-step a move of unit length in the $xy$-plane in the direction $2\pi\alpha$ (measured counterclockwise with respect to the horizontal). For example, if you are located at the origin, a $(1/2)$-step (corresponding to an angle of $\pi$, or 180 degrees) will put you at position $(-1, 0)$, a $(1/3)$-step (120 degrees) will place you at the point $(-1/2, \sqrt{3}/2)$, a $(1/4)$-step you will place you at $(0, 1)$, and after performing all three of these steps, you will be located at $(-1 + (-1/2) + 0, 0 + \sqrt{3}/2 + 1) = (-3/2, 1 + \sqrt{3}/2)$.*

   *Suppose you start at the origin and perform a sequence of $(p/q)$-steps, where $p$ and $q$ range over all pairs of integers $(p, q)$ with $1 \leq p < q \leq 2006$, giving a total of $2005 \cdot 2006/2 = 2011015$ steps of unit length. Where will you be located at the end of these 2011015 steps?*

   **Solution.** At first glance, this problem looks impossibly difficult, but it becomes doable when approached the right way.

   If we think of the $xy$-plane as the complex plane with a point $(x, y)$ corresponding to the complex number $z = x+iy$, then moving by an $\alpha$-step corresponds to adding $e^{2\pi i\alpha}$ to the complex number corresponding to your current location. Thus, the location after performing the given sequence of moves is the determined by the sum

$$S = \sum_{q=2}^{2006} \sum_{p=1}^{q-1} e^{2\pi i \frac{p}{q}},$$

   so it remains to evaluate this double sum. To this end note that each of the inner sums here is a finite geometric series which is easily evaluated:

$$\sum_{p=1}^{q-1} \left(e^{2\pi i/q}\right)^p = \frac{e^{2\pi i(q/q)} - e^{2\pi i/q}}{e^{2\pi i/q} - 1} = -1.$$

   Hence, for each value of $q$, the corresponding inner sum contributes an amount $-1$ to $S$, and since there are 2005 values of $q$, we get $S = -2005$. This corresponds to the location $(-2005, 0)$ in the $xy$-plane.