UIUC UNDERGRADUATE MATH CONTEST
April 8, 2006

1. Determine, without numerical calculations, which of the two numbers \( \sqrt{2005} \sqrt{2006} \) and \( \sqrt{2006} \sqrt{2005} \) is larger.

2. Let \( f(x) = e^{x^2} \sin x \). Find, with proof, \( f^{(2006)}(0) \), the 2006th derivative of \( f \) at 0.

3. Evaluate the series
\[
\sum_{n=0}^{\infty} \frac{1}{2006^{2n} - 2006^{-2n}} = \frac{1}{2006^1 - 2006^{-1}} + \frac{1}{2006^2 - 2006^{-2}} + \frac{1}{2006^4 - 2006^{-4}} + \cdots
\]
and express it as a rational number.

4. For any positive integer \( n \), define a sequence \( \{n_k\}_{k=0}^{\infty} \) as follows: Set \( n_0 = n \), and for each \( k \geq 1 \), let \( n_k \) be the sum of the (decimal) digits of \( n_{k-1} \). For example, for \( n = 1729 \) we get the sequence 1729, 19, 10, 1, 1, 1, \ldots. In general, for any given starting value \( n \), the resulting sequence \( \{n_k\} \) eventually stabilizes at a single digit value. Let \( f(n) \) denote this value; for example, \( f(1729) = 1 \). Determine \( f(2^{2006}) \).

5. Let \( D = \{d_1, d_2, \ldots, d_{10}\} \) be a set of 10 distinct positive integers. Show that any sequence of 2006 integers from \( D \) contains a block of one or more consecutive terms whose product is the square of a positive integer.

6. Given a real number \( \alpha \) with \( 0 \leq \alpha < 1 \), define an \( \alpha \)-step a move of unit length in the \( xy \)-plane in the direction \( 2\pi \alpha \) (measured counterclockwise with respect to the horizontal). For example, if you are located at the origin, a \( (1/2) \)-step (corresponding to an angle of \( \pi \), or 180 degrees) will put you at position \((-1,0)\), a \((1/3)\)-step (120 degrees) will place you at the point \((-1/2, \sqrt{3}/2)\), a \((1/4)\)-step you will place you at \((0,1)\), and after performing all three of these steps, you will be located at \((-1 + (-1/2) + 0, 0 + \sqrt{3}/2 + 1) = (-3/2, 1 + \sqrt{3}/2)\).

Suppose you start at the origin and perform a sequence of \((p/q)\)-steps, where \( p \) and \( q \) range over all pairs of integers \((p,q)\) with \( 1 \leq p < q \leq 2006 \), giving a total of \( 2005 \cdot 2006/2 = 2011015 \) steps of unit length. Where will you be located at the end of these 2011015 steps?

[Solutions at http://www.math.uiuc.edu/contests.html]