UIUC UNDERGRADUATE MATH CONTEST
April 8, 2006

1. *Determine*, without numerical calculations, *which of the two numbers* $\sqrt{2005}^{\sqrt{2006}}$ *and* $\sqrt{2006}^{\sqrt{2005}}$ *is larger.*

2. *Let* $f(x) = e^{x^2}\sin x$. *Find, with proof,* $f^{(2006)}(0)$, *the* $2006th$ *derivative of* f *at* 0.

3. *Evaluate the series*
$$\sum_{n=0}^{\infty}\frac{1}{2006^{2^n}-2006^{-2^n}}=\frac{1}{2006^1-2006^{-1}}+\frac{1}{2006^2-2006^{-2}}+\frac{1}{2006^4-2006^{-4}}+\cdots$$
and express it as a rational number.

4. *For any positive integer* n, *define a sequence* $\{n_k\}_{k=0}^{\infty}$ *as follows: Set* $n_0 = n$, *and for each* $k \geq 1$, *let* n_k *be the sum of the (decimal) digits of* n_{k-1}. *For example, for* $n = 1729$ *we get the sequence* $1729, 19, 10, 1, 1, 1, \ldots$. *In general, for any given starting value* n, *the resulting sequence* $\{n_k\}$ *eventually stabilizes at a single digit value. Let* $f(n)$ *denote this value; for example,* $f(1729) = 1$. *Determine* $f(2^{2006})$.

5. *Let* $D = \{d_1, d_2, \ldots, d_{10}\}$ *be a set of* 10 *distinct positive integers. Show that any sequence of* 2006 *integers from* D *contains a block of one or more consecutive terms whose product is the square of a positive integer.*

6. *Given a real number* α *with* $0 \leq \alpha < 1$, *define an* α*-step a move of unit length in the* xy*-plane in the direction* $2\pi\alpha$ *(measured counterclockwise with respect to the horizontal). For example, if you are located at the origin, a* $(1/2)$*-step (corresponding to an angle of* π, *or* 180 *degrees) will put you at position* $(-1, 0)$, *a* $(1/3)$*-step (*120 *degrees) will place you at the point* $(-1/2, \sqrt{3}/2)$, *a* $(1/4)$*-step you will place you at* $(0, 1)$, *and after performing all three of these steps, you will be located at* $(-1 + (-1/2) + 0, 0 + \sqrt{3}/2 + 1) = (-3/2, 1 + \sqrt{3}/2)$.

   Suppose you start at the origin and perform a sequence of (p/q)*-steps, where* p *and* q *range over all pairs of integers* (p, q) *with* $1 \leq p < q \leq 2006$, *giving a total of* $2005 \cdot 2006/2 = 2011015$ *steps of unit length. Where will you be located at the end of these* 2011015 *steps?*

[Solutions at `http://www.math.uiuc.edu/contests.html`]