UIUC UNDERGRADUATE MATH CONTEST
April 16, 2005
Solutions

1. **For which positive integers n does the equation**

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$$

have a solution in integers $a_i = \pm 1$? Explain!

Solution. If n is divisible by 4, then letting $a_1, a_2, \ldots, a_n$ be the pattern $(1, 1, -1, -1)$ repeated $n/4$ times, the terms a_ia_{i+1} in the above sum are alternately 1 and -1, and their sum is equal to 0. Thus, for all n divisible by 4, the given equation has a solution.

We now show that if n is not divisible by 4, there is no solution in integers ± 1. This is obvious in the case n is odd, since then the left-hand side of the equation consists of a sum of an odd number of terms, each ± 1, and thus cannot be equal to 0.

It remains to consider the case when $n = 2m$, where m is odd. Suppose there exist integers $a_i = \pm 1$, $i = 1, 2, \ldots, n$, for which the above equation holds. Set $a_{n+1} = a_1$, so that the equation can be written as $\sum_{i=1}^{n} a_ia_{i+1} = 0$. Since $n = 2m$ and each of the terms a_ia_{i+1} is ± 1, exactly m of these terms be equal to 1, and m must be equal to -1. Hence the product of all $2m$ terms must be equal to $(1)^m(-1)^m = -1$, since m was assumed to be odd. On the other hand, a direct calculation shows that the product is equal to

$$\prod_{i=1}^{n} a_ia_{i+1} = \prod_{i=1}^{n} a_i^2 = 1,$$

so we have reached a contradiction. Thus, no solution exists when $n = 2m$ with m odd.

2. **Evaluate the integral $I = \int_0^{\pi} \ln(\sin x)dx$.**

Solution. We will show that $I = -2\pi \ln 2$.

Using the identity $\sin x = 2\sin(x/2)\cos(x/2)$ we can write I as

$$\begin{aligned}
I &= \int_0^\pi \ln 2 dx + \int_0^\pi \ln\sin(x/2)dx + \int_0^\pi \ln\cos(x/2)dx \\
&= \pi\ln 2 + 2\int_0^{\pi/2} \ln\sin y dy + 2\int_0^{\pi/2} \ln\cos y dy \\
&= \pi\ln 2 + 2I_1 + 2I_2,
\end{aligned}$$

say. Setting $y = \pi/2 - u$ and using the relation $\cos(\pi/2 - u) = \sin u$, we see that $I_1 = I_2$, and since $\sin x = \sin(\pi - x)$, we have also

$$2I_1 = \int_0^{\pi/2} (\ln\sin x + \ln\sin(\pi - x))dx = \int_0^\pi \ln\sin x dx = I.$$

Hence the above relation implies $I = \pi\ln 2 + 4I_1 = \pi\ln 2 + 2I$, and solving for I gives $I = -\pi\ln 2$, as claimed.

3. **Suppose 3 players, P_1, P_2, P_3, seated at a round table, take turns rolling a die. Player P_1 rolls first, followed by P_2, etc. Once a player has rolled a 6, the game is stopped and that player is declared the winner. If no 6 has been obtained after each of P_1, P_2, P_3 have rolled the die once, player P_1 gets to roll again, followed by P_2, etc. Find the probability that the first player, P_1, wins this game.**

   Solution. Let N denote the number of "rounds" in the game, i.e., the number of rolls needed until a six shows up (including the roll at which the six shows up). Then the first player wins if and only if N is equal to one of the values $1, 4, 7, \dots$, i.e., if N is of the form $N = 3k+1$, $k = 0, 1, \dots$. Now, $N = n$ for a given value n if and only no 6 is rolled in the first $n-1$ rolls and a 6 appears in the n-th roll; the probability for this event is $p_n = (5/6)^{n-1}(1/6)$. Hence the probability that player P_1 wins is

$$\begin{aligned}
\sum_{k=0}^\infty p_{3k+1} &= \sum_{k=0}^\infty (5/6)^{3k+1-1}(1/6) \\
&= \frac{1}{6}\sum_{k=0}^\infty ((5/6)^3)^k = \frac{1}{6(1-(5/6)^3)} = \frac{36}{91}.
\end{aligned}$$

4. **Prove that, for any real numbers x and y in the interval $(0, 1)$,**

$$\left(\frac{x+y}{2}\right)^{x+y} \le x^x y^y.$$

Solution. Let $f(x) = \log x^x = x \log x$. Taking logarithms and dividing both sides by 2, the given inequality is equivalent to

$$f\left(\frac{1}{2}(x+y)\right) \le \frac{1}{2}(f(x) + f(y)). \tag{1}$$

Now note that $f'(x) = \log x + 1$, and $f''(x) = 1/x > 0$ for positive x, so the function $f(x)$ is convex (i.e., concave up) for $x > 0$. Since any convex function satisfies (1), the result follows. (The fact that (1) holds for any convex function f is easily seen by sketching the graph of a convex function and comparing the value of the function at the midpoint of an interval $[x, y]$, with the average of the values of the function at the two end points x and y.)

5. **Determine, with proof, whether the series**

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.8+\sin n}}$$

converges or diverges.

Solution. We show that the series diverges. Note that $\sin x \le -\sqrt{3}/2$ whenever x falls into one of the intervals

$$I_k = [(2k + 4/3)\pi, (2k + 5/3)\pi], \quad k = 0, \pm 1, \pm 2, \dots$$

Each of these intervals has length $\pi/3 > 1$ and the gap between two successive intervals has length $< (5/3)\pi < 6$. Hence, among any 7 consecutive integers n at least one must fall into one of the intervals I_k; for this value of n we have $1.8 + \sin n < 1.8 - \sqrt{3}/2 < 1$, so the corresponding term in the above series is greater than $1/n$. Therefore the above series is bounded from below by

$$\sum_{m=0}^{\infty} \sum_{n=7m+1}^{7m+7} \frac{1}{n^{1.8+\sin n}} \ge \sum_{m=0}^{\infty} \frac{1}{7m+7} = \frac{1}{7} \sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

and hence diverges.

6. **Let $a_1, \dots, a_n$ be a set of positive integers such that the product $\prod_{i=1}^{n} a_i$ has fewer than n distinct prime divisors. Prove that there exists a nonempty subset $I \subset \{1, \dots, n\}$ such that $\prod_{i \in I} a_i$ is a perfect square.**

Solution. Let $p_1, \dots, p_m$ denote the prime divisors of $\prod_{i=1}^{n} a_i$. Then each a_i can be written as $a_i = \prod_{j=1}^{m} p_j^{\alpha_{ij}}$, with nonnegative integers α_{ij}, and for any subset $I \subset \{1, 2, \dots, n\}$, the product $\prod_{i \in I} a_j$ has prime factorization $\prod_{j=1}^{m} p^{\alpha_{Ij}}$, with $\alpha_{Ij} = \sum_{i \in I} \alpha_{ij}$. Such a product is a perfect square if and only if the exponents α_{Ij}, $j = 1, \ldots, m$, are all even, i.e., if and only if the system of congruences

$$\sum_{i \in I} \alpha_{ij} \equiv 0 \bmod 2, \quad j = 1, \ldots, m \tag{1}$$

holds. Setting $\epsilon_i = 1$ if $i \in I$, and $\epsilon_i = 0$ otherwise, (1) can be written as

$$\sum_{i=1}^{n} \alpha_{ij} \epsilon_i \equiv 0 \bmod 2, \quad j = 1, \ldots, m \tag{2}$$

Thus, we need to show that, if $m < n$, then the latter system has a solution $\epsilon_i \in \{0, 1\}$, with $\epsilon_i \neq 0$.

To this end we consider first the system

$$\sum_{i=1}^{n} \alpha_{ij} x_i = 0, \quad j = 1, \ldots, m, \tag{3}$$

in the variables $x_1, \ldots, x_n$. This is a system of m linear equations with integer coefficients in n variables. Since, by assumption, $m < n$, by elementary linear algebra this system has a solution in integers $x_1, \ldots, x_n$ that are not all zero. Dividing through by the greatest common divisor, we may further assume that the x_i's do not have a common prime divisor and, in particular, are not all even. Hence, defining $\epsilon_i = 1$ if x_i is even, and $\epsilon_i = 0$ otherwise, and reducing both sides of (3) modulo 2, we obtain a nontrivial solution $(\epsilon_1, \ldots, \epsilon_n)$ to (2).