UIUC UNDERGRADUATE MATH CONTEST
April 16, 2005

1. **For which positive integers n does the equation**

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$$

have a solution in integers $a_i = \pm 1$? Explain!

2. **Evaluate the integral $I = \int_0^\pi \ln(\sin x)dx$.**

3. **Suppose 3 players, P_1, P_2, P_3, seated at a round table, take turns rolling a die. Player P_1 rolls first, followed by P_2, etc. Once a player has rolled a 6, the game is stopped and that player is declared the winner. If no 6 has been obtained after each of P_1, P_2, P_3 have rolled the die once, player P_1 gets to roll again, followed by P_2, etc. Find the probability that the first player, P_1, wins this game.**

4. **Prove that, for any real numbers x and y in the interval $(0, 1)$,**

$$\left(\frac{x+y}{2}\right)^{x+y} \leq x^x y^y.$$

5. **Determine, with proof, whether the series**

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.8+\sin n}}$$

converges or diverges.

6. **Let $a_1, \ldots, a_n$ be a set of positive integers such that the product $\prod_{i=1}^n a_i$ has fewer than n distinct prime divisors. Prove that there exists a nonempty subset $I \subset \{1, \ldots, n\}$ such that $\prod_{i \in I} a_i$ is a perfect square.**