UIUC UNDERGRADUATE MATH CONTEST
April 17, 2004
Solutions

1. Suppose a, b and c are integers such that the equation $ax^2+bx+c=0$ has a rational solution. Prove that at least one of the integers a, b and c must be even.

   Solution. We argue by contradiction. Suppose a, b, and c are all odd and that $x=p/q$, with $(p,q)=1$, is a rational solution of $ax^2+bx+c=0$. Clearing denominators, we obtain $(*)$ $ap^2+bpq+cq^2=0$. Since we assumed p and q are relatively prime, they cannot be both even. If p and q are both odd, then, in view of our initial assumption that a, b and c are odd, each term on the left of $(*)$ is odd, so the left-hand side is odd and we have a contradiction. If exactly one of p and q is odd, then exactly two of the three terms on the left of $(*)$ are even, and so the left-hand side is odd and we again arrive at a contradiction. Thus, a contradiction arises in either case, and a, b, and c therefore cannot all be odd.

2. Let F_n denote the Fibonacci sequence, defined by $F_1=1$, $F_2=1$, and $F_{n+1}=F_n+F_{n-1}$ for $n\ge 2$. Evaluate

   $$\sum_{k=1}^{\infty}\frac{F_k}{3^k}.$$

   Solution. First note that, by induction, we have $F_n\le 2^n$ for all n, so the given series is majorized by the geometric series $\sum_{k=1}^{\infty}(2/3)^k$, and hence is (absolutely) convergent. Let S denote the sum of this series. Using the Fibonacci recurrence for terms with $k\ge 3$, we obtain

   $$\begin{aligned} S &= \frac{F_1}{3}+\frac{F_2}{3^2}+\sum_{k=3}^{\infty}\frac{F_{k-1}}{3^k}+\sum_{k=3}^{\infty}\frac{F_{k-2}}{3^k} \\ &= \frac{1}{3}+\frac{1}{9}+\frac{1}{3}\sum_{k=2}^{\infty}\frac{F_k}{3^k}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{F_k}{3^k} \\ &= \frac{1}{3}+\frac{1}{9}+\frac{1}{3}\left(S-\frac{1}{3}\right)+\frac{1}{9}S. \end{aligned}$$

Solving for S we get

$$S = \frac{\frac{1}{3}}{1 - \frac{1}{3} - \frac{1}{9}} = \frac{3}{5}.$$

3. Define a sequence $\{a_n\}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and

$$a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1$$

for $n \geq 3$. Find, with proof, a_{2004}.

Solution. Let $b_n = a_n - a_{n-1}$. From the given recurrence for a_n we obtain

$$b_n = b_{n-2} + 1 \quad (n \geq 3)$$

with initial conditions $b_1 = 1$, $b_2 = 1$. This implies, by induction, $b_{2n} = b_{2n-1} = n$ for all $n \geq 1$. Hence

$$\begin{aligned} a_{2n} &= a_0 + \sum_{k=1}^{n}(a_{2k} - a_{2k-2}) = \sum_{k=1}^{n}(b_{2k} + b_{2k-1}) \\ &= \sum_{k=1}^{n}(2k) = 2\frac{n(n+1)}{2} = n(n+1). \end{aligned}$$

Hence $a_{2004} = 1002 \cdot 1003 = 1005006$.

4. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where the a_k are real numbers. Suppose that $f(x)$ satisfies $|f(x)| \leq |\sin x|$ for all real x. Show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

Solution. We have $f'(x) = \sum_{k=1}^{n} ka_k \cos kx$, and so $f'(0) = \sum_{k=1}^{n} ka_k$. Thus, the claim is equivalent to $|f'(0)| \leq 1$. Now, by the definition of the derivative, we have $f'(0) = \lim_{x\to 0} f(x)/x$, and since $|f(x)| \leq |\sin x| \leq |x|$ for all x, the inequality $|f'(0)| \leq 1$ follows.

5. Let

$$f(n) = \sum_{k=1}^{n} \left[\frac{n}{k}\right],$$

where $[x]$ denotes the greatest integer $\leq x$, and let $g(n) = (-1)^{f(n)}$. Find, with proof, $g(2004)$.

Solution. Note that $g(n) = 1$ or $g(n) = -1$ depending on whether $f(n)$ is even or odd. Since, for each k, $[n/k]$ counts the number of positive integers m for which $km \leq n$, the function $f(n)$ is equal to the number of pairs (k, m) of positive integers that satisfy $km \leq n$. Among these pairs, the number of those with $k \neq m$ is even since we can pair up (k, m) with (m, k). Hence, modulo 2, $f(n)$ is congruent to the number of remaining pairs in the above count, i.e., those of the form (k, k) with $k \leq n$. Clearly, there are $[\sqrt{n}]$ such k, so we have $f(n) \equiv [\sqrt{n}]$ modulo 2, and therefore $g(n) = (-1)^{f(n)} = (-1)^{[\sqrt{n}]}$. Since $44^2 = 1936$ and $45^2 = 2025$, we have $[\sqrt{2004}] = 44$, so $g(2004) = (-1)^{44} = 1$.

6. Find, with proof, **all** functions $f(x)$ that are defined for real numbers x with $|x| < 1$, continuous at $x = 0$, and which satisfy

$$f(0) = 1, \quad f(x^2) = \frac{f(x)}{1+x} \quad (|x| < 1).$$

Solution. First note that the function $f(x) = 1/(1-x)$ satisfies the given conditions. We will show that this is the only solution. Suppose $f(x)$ is a solution, and let $g(x) = (1-x)f(x)$. Note first that since, by assumption, $f(x)$ is continuous at $x = 0$, $g(x)$ is also continuous at $x = 0$. From the given relation for $f(x)$ we obtain $g(0) = f(0) = 1$, and for $|x| < 1$,

$$g(x^2) = (1-x^2)f(x^2) = \frac{(1-x^2)f(x)}{1+x} = g(x).$$

Iterating this identity, we get

$$g(x) = g(x^{2^n})$$

for any positive integer n. Since, for $|x| < 1$, x^{2^n} tends to 0 as $n \to \infty$ and g is continuous at 0, it follows that

$$g(x) = \lim_{n\to\infty} g(x^{2^n}) = g(0) = 1$$

for all x with $|x| < 1$. Hence $f(x) = g(x)/(1-x) = 1/(1-x)$, so the function $1/(1-x)$ is indeed the only solution.