UIUC UNDERGRADUATE MATH CONTEST
April 17, 2004

1. Suppose a, b and c are integers such that the equation $ax^2+bx+c=0$ has a rational solution. Prove that at least one of the integers a, b and c must be even.

2. Let F_n denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Evaluate
$$\sum_{k=1}^{\infty} \frac{F_k}{3^k}.$$

3. Define a sequence $\{a_n\}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and
$$a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1$$
for $n \geq 3$. Find, with proof, a_{2004}.

4. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where the a_k are real numbers. Suppose that $f(x)$ satisfies $|f(x)| \leq |\sin x|$ for all real x. Show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

5. Let
$$f(n) = \sum_{k=1}^{n} \left[\frac{n}{k}\right],$$
where $[x]$ denotes the greatest integer $\leq x$, and let $g(n) = (-1)^{f(n)}$. Find, with proof, $g(2004)$.

6. Find, with proof, **all** functions $f(x)$ that are defined for real numbers x with $|x| < 1$, continuous at $x = 0$, and which satisfy
$$f(0) = 1, \quad f(x^2) = \frac{f(x)}{1+x} \quad (|x| < 1).$$