UIUC UNDERGRADUATE MATH CONTEST
April 12, 2003
Solutions

1. Let
$$N = 9 + 99 + 999 + \cdots + \overbrace{99\ldots9}^{99}.$$
Determine the sum of digits of N.

Solution. The answer is 99. To see this, evaluate N explicitly as follows:
$$\begin{aligned} N &= (10-1) + (100-1) + \cdots + (\overbrace{10\ldots0}^{100} - 1) \\ &= \overbrace{11\ldots1}^{99}0 - 99 = \overbrace{11\ldots1}^{97}011. \end{aligned}$$

2. Evaluate
$$\frac{1}{1\cdot2\cdot3} + \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} + \cdots$$

Solution. Let S be the sum of the given series (which is easily seen to be convergent, e.g., by comparing it with the series $\sum_{n=1}^{\infty} n^{-2}$). We will show that $S = \ln 2 - 1/2$. Denoting by S_N the N-th partial sum of this series, we have
$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{2n(2n-1)(2n+1)} = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{2n}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{2n-1} - 2\frac{1}{2n} + \frac{1}{2n+1}\right) \end{aligned}$$

Letting $U_N = \sum_{n=1}^{N} 1/(2n)$ and $V_N = \sum_{n=1}^{N} 1/(2n-1)$ denote the partial sums over the even resp. odd terms in the harmonic series, we can write the last expression as
$$S_N = \frac{1}{2}\left(V_N - 2U_N + V_N - 1 + \frac{1}{2N+1}\right) = V_N - U_N - \frac{1}{2} + \frac{1}{2(2N+1)}.$$

But

$$V_N - U_N = \sum_{n=1}^{2N} \frac{(-1)^{n+1}}{n},$$

which converges to $\ln 2$ as $N \to \infty$. Hence $S = \lim_{N\to\infty} S_N = \ln 2 - 1/2$, as claimed.

3. Prove that the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \prod_{i=1}^{n} \left(n^2 + i^2\right)^{1/n}$$

exists and find its value.

Solution. We will show that the limit is equal to $2e^{-2+\pi/2}$. Let $P_n = n^{-2}\prod_{i=1}^{n}(n^2+i^2)^{1/n}$. Factoring out $(n^2)^{1/n}$ from each term in the product, we see that $P_n = \prod_{i=1}^{n}(1+(i/n)^2)^{1/n}$, and hence

$$\log P_n = \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + \left(\frac{i}{n}\right)^2\right).$$

The term on the right is a Riemann sum for the integral $I = \int_0^1 \log(1+x^2)dx$, and therefore converges to this integral as $n \to \infty$, i.e., we have $\lim_{n\to\infty} \log P_n = I$. Hence the limit $\lim_{n\to\infty} P_n$ exists, and is equal to e^I. It remains to evaluate the integral I. This is a routine exercise in integration by parts:

$$\begin{aligned} I &= x\log(1+x^2)\Big|_0^1 - \int_0^1 \frac{x(2x)}{1+x^2}dx \\ &= \log 2 - \int_0^1 \left(2 - \frac{2}{1+x^2}\right) dx \\ &= \log 2 - 2 + 2\arctan x\Big|_0^1 = \log 2 - 2 + \frac{\pi}{2}. \end{aligned}$$

4. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers, and let b_n be the arithmetic mean of $a_1, a_2, \ldots, a_n$. Prove that if $\sum_{n=1}^{\infty} 1/a_n$ converges, then so does $\sum_{n=1}^{\infty} 1/b_n$.

Solution. Let $S_i = \sum_{2^i \le n < 2^{i+1}} a_n$. Then, for each $i \ge 0$ and $2^{i+1} \le n < 2^{i+2}$,

$$b_n = \frac{1}{n}\sum_{k=1}^{n} a_k \ge \frac{S_i}{2^{i+2}},$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \le \frac{1}{b_1} + \sum_{i=0}^{\infty} \sum_{2^{i+1} \le n < 2^{i+1}} \frac{2^{i+2}}{S_i} = \frac{1}{b_1} + 4\sum_{i=0}^{\infty} \frac{2^{2i}}{S_i}.$$

Now, $S_i/2^i$ is the arithmetic mean of the 2^i numbers a_n, $2^i \le n < 2^{i+1}$. By the arithmetic-harmonic mean inequality, this is at least equal to the harmonic mean of these numbers, namely $2^i \left(\sum_{2^i \le n < 2^{i+1}} 1/a_n\right)^{-1}$. Hence,

$$\frac{2^{2i}}{S_i} \le \sum_{2^i \le n < 2^{i+1}} \frac{1}{a_n},$$

and it follows that

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \le \frac{1}{b_1} + 4\sum_{i=0}^{\infty} \frac{1}{a_n}.$$

Thus, the convergence of $\sum_{n=1}^{\infty} 1/a_n$ implies that of $\sum_{n=1}^{\infty} 1/b_n$.

5. Is it possible to arrange the numbers $1, 2, \ldots, 2003$ in a row so that each number, with the exception of the two numbers at the left and right end, is either the sum or the absolute value of the difference of the two numbers surrounding it?

   Solution. The answer is no, as can be seen by considering the parity of the numbers. The given condition implies that, except for the two integers at the left and right ends of the row, an integer n in the row must be surrounded by two odd integers or two even integers in case n is even, and by an odd and an even integer, in case n is odd. Thus, after reducing the numbers in the row modulo 2, the only blocks of three that can occur are 000, 011, 101, and 110. It follows that the entire row is determined (modulo 2) by its first two elements: If the row starts with 00, all elements in the row must be 0; if it starts with 01, the row is of the form $011011011\ldots$; if it starts with 10, it is of the form $101101101\ldots$; and if it starts with 11, it must be of the form $110110110\ldots$. In the first case, all numbers would have to be even, and in the other cases at least $2[2003/3] = 1334$ of the numbers would have to be odd. Since exactly 1002 of the given numbers are odd, we obtain a contradiction in all of these cases.

6. Find the smallest integer $n > 11$ for which there is a polynomial of degree n with the following properties:

   (a) $P(k) = k^{11}$ for $k = 1, 2, \ldots, n$;

   (b) $P(0)$ is an integer;

   (c) $P(-1) = 2003$.

   Solution. The answer is $n = 166$. To see this, suppose first that $P(x)$ is a polynomial of degree n satisfying the three conditions (a), (b), and (c). Consider the polynomial $Q(x) = P(x) - x^{11}$. Then $Q(x)$ has degree at most n, and condition (a) implies that $Q(x)$ has a root at each of the numbers $k = 1, 2, \ldots, n$. It follows that $Q(x)$ is of the form $Q(x) = C(x-1)(x-2)\ldots(x-n)$ for some constant C. To determine C, we use condition (c), which implies

$$2003 = P(-1) = Q(-1) + (-1)^{11} = C(-1)^n(n+1)! - 1.$$

Hence $C = 2004(-1)^n/(n+1)!$. Now,

$$P(0) = Q(0) = C(-1)^n n! = \frac{2004}{n+1},$$

so condition (b) holds if and only if $n+1$ is a divisor of 2004. Thus, any polynomial $P(x)$ of degree n satisfying all three conditions (a)–(c) is of the form $P(x) = C(x-1)(x-2)\ldots(x-n) + x^{11}$ with C as above and $n+1$ a divisor of 2004. Conversely, it is easy to see that any polynomial of this form satisfies (a)–(c). Therefore the number n sought in the problm is the smallest number $n > 11$ such that $n+1$ divides 2004, i.e., n is 1 less than the smallest divisor of 2004 that exceeds 13. Factoring 2004, we get $2004 = 2^2 \cdot 3 \cdot 167$. Hence, the smallest divisor of 2004 exceeding 13 is 167, and so $n = 166$.