UIUC UNDERGRADUATE MATH CONTEST
April 12, 2003

1. Let
$$N = 9 + 99 + 999 + \cdots + \overbrace{99\ldots9}^{99}.$$
Determine the sum of digits of N.

2. Evaluate
$$\frac{1}{1\cdot2\cdot3} + \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} + \cdots$$

3. Prove that the limit
$$\lim_{n\to\infty} \frac{1}{n^2} \prod_{i=1}^{n} \left(n^2 + i^2\right)^{1/n}$$
exists and find its value.

4. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers, and let b_n be the arithmetic mean of $a_1, a_2, \ldots, a_n$. Prove that if $\sum_{n=1}^{\infty} 1/a_n$ converges, then so does $\sum_{n=1}^{\infty} 1/b_n$.

5. Is it possible to arrange the numbers $1, 2, \ldots, 2003$ in a row so that each number, with the exception of the two numbers at the left and right end, is either the sum or the absolute value of the difference of the two numbers surrounding it?

6. Find the smallest integer $n > 11$ for which there is a polynomial of degree n with the following properties:
   (a) $P(k) = k^{11}$ for $k = 1, 2, \ldots, n$;
   (b) $P(0)$ is an integer;
   (c) $P(-1) = 2003$.